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Revenue maximization in the dynamic knapsack problem

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We analyze maximization of revenue in the dynamic and stochastic knapsack problem where a given capacity needs to be allocated by a given deadline to sequentially arriving agents. Each agent is described by a two-dimensional type that reflects his capacity requirement and his willingness to pay per unit of capacity. Types are private information. We first characterize implementable policies. Then we solve the revenue maximization problem for the special case where there is private information about per-unit values, but capacity needs are observable. After that we derive two sets of additional conditions on the joint distribution of values and weights under which the revenue maximizing policy for the case with observable weights is implementable, and thus optimal also for the case with two-dimensional private information. In particular, we investigate the role of concave continuation revenues for implementation. We also construct a simple policy for which per-unit prices vary with requested weight but not with time, and we prove that it is asymptotically revenue maximizing when available capacity and time to the deadline both go to infinity. This highlights the importance of nonlinear as opposed to dynamic pricing.

Keywords. Knapsack, revenue maximization, dynamic mechanism design.

JEL classification. D42, D44, D82.

1. Introduction

The knapsack problem is a classic combinatorial optimization problem with numerous practical applications: several objects with given, known capacity needs (or weights) and given, known values must be packed into a "knapsack" of given capacity to maximize the total value of the included objects. In the dynamic and stochastic version (see

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Ross and Tsang 1989), objects sequentially arrive over time and their weight–value combination is stochastic but becomes known to the designer at arrival times. Objects cannot be recalled later, so it must be decided upon arrival whether an object is included or not. Several applications that come to mind are logistic decisions in the freight transportation industry, the allocation of fixed capacities in the travel and leisure industries (e.g., airlines, trains, hotels, rental cars), the allocation of fixed equipment or personnel in a given period of time (e.g., equipment and personnel for medical procedures in an emergency), the allocation of fixed budgets to investment opportunities that appear sequentially, the allocation of research and development funds to emerging ideas, and the allocation of dated advertising space on web portals.

In the present paper we add incomplete information to the dynamic and stochastic setting. In this way, we obtain a dynamic monopolistic screening problem: there is a finite number of periods, and at each period a request for capacity arrives from an agent who is impatient and privately informed about both his valuation per unit of capacity and the needed capacity.[1] Each agent derives positive utility if he gets the needed capacity (or more), and zero utility otherwise. The designer accepts or rejects the requests so as to maximize the revenue obtained from the allocation.

The dynamic and stochastic knapsack problem with complete information about values and requests was analyzed by Papastavrou et al. (1996) and by Kleywegt and Papastavrou (2001). These authors characterize optimal policies in terms of weight-dependent value thresholds. Kincaid and Darling (1963) and Gallego and van Ryzin (1994) look at a model that can be reinterpreted as having (one-dimensional) incomplete information about values, but in their frameworks all requests have the same known weight.[2] In particular, Gallego and van Ryzin show that optimal revenue is concave in capacity in the case of equal weights. Kleywegt and Papastavrou give examples showing that total value is not necessarily globally concave in capacity if the weight requests are heterogeneous, and provide a sufficient condition for this structural property to hold. Gallego and van Ryzin also show that the optimal policy, which exhibits complicated time dynamics, can often be replaced by a simple time-independent policy without much loss: the simple policy performs asymptotically optimal as the number of periods and the units to be sold go to infinity. Finally, Gershkov and Moldovanu (2009) generalize the Gallego–van Ryzin model to incorporate objects with the same weight but with several qualities that are equally ranked by all agents, independently of their types (which are also one dimensional).

The theory of multidimensional mechanism design is relatively complex: the main problem is that incentive compatibility—which in the one-dimensional case often reduces to a monotonicity constraint—imposes, besides a monotonicity requirement, an integrability constraint that is not easily included in maximization problems (see examples in Rochet 1985, Armstrong 1996, Jehiel et al. 1999, and the survey of Rochet and Stole 2003). Our implementation problem is special though because useful deviations in

[1] Our results are easily extended to the setting where arrivals are stochastic and/or time is continuous.

[2] We refer the reader to the book by Talluri and Van Ryzin (2004) for references to the large literature on revenue (or yield) management that adopts variations on these models.

the weight dimension can only be one-sided (upward). This feature allows a less cumbersome characterization of implementable policies that can be embedded in the dynamic analysis under certain conditions on the joint distribution of values and weights of the arriving agents. Other multidimensional mechanism design problems with restricted deviations in one or more dimensions have been studied by Blackorby and Szalay (2007), Che and Gale (2000), Iyengar and Kumar (2008), Kittsteiner and Moldovanu (2005), and Pai and Vohra (2009).

1.1 *Outline and preview of results*

We first characterize implementable policies, as explained above. Then we solve the revenue maximization problem for the case where there is private information about per-unit values, but weights are observable. We will sometimes refer to this as the relaxed problem. Under a standard monotonicity assumption on virtual values, this is the virtual value analog of the problem solved by Papastravou, Rajagopalan, and Kleywegt. The resulting optimal policy is Markovian, is deterministic, and has a threshold property with respect to virtual values. It is important to emphasize that this policy need not be implementable for the case where both values and weights are unobservable, unless additional conditions are imposed. Our main results in the first part of the paper are therefore concerned with the implementability of the relaxed optimal solution: we derive two sets of additional conditions on the joint distribution of values and weights under which the revenue maximizing policy for the case with observable weights is implementable, and thus optimal also for the case with two-dimensional private information. The first condition—which is satisfied in a variety of intuitive settings—is a hazard rate ordering that expresses a form of positive correlation between weights and values. It ensures that the incentive constraint in the capacity dimension is never binding. Related conditions can be found in previous work on multidimensional mechanism design with restricted deviations mentioned above, e.g., in the papers of Pai and Vohra, Iyengar and Kumar, or Blackorby and Szalay. More interestingly, we also draw a connection between incentive compatibility and the structural property of concavity of revenue in capacity. Concavity of optimal revenue in the relaxed problem creates a tendency to set higher virtual value thresholds for higher capacity requests. It is then less attractive for agents to overstate their capacity needs, which facilitates the implementation of the relaxed solution by relaxing the incentive constraints. We quantify this relation in our second set of additional conditions: concavity of revenue combined with a (substantial) weakening of the hazard rate order imply implementability of the relaxed solution. For completeness, we also briefly translate to our model the sufficient condition for concavity of revenue due to Papastavrou, Rajagopalan, and Kleywegt so as to obtain a condition on the model's primitives.

In the second part of the paper we construct—for general distributions of weights and values—a time-independent, nonlinear price schedule that is asymptotically revenue maximizing when the available capacity and the time to the deadline both go to infinity, and when weights are observable. This extends an asymptotic result by Gallego and van Ryzin (for a detailed discussion, see Section 5) and suggests that complicated dynamic pricing may not be that important for revenue maximization if the distribution of agents' types is known. Our result emphasizes though that nonlinear pricing remains asymptotically important in dynamic settings. As a nice link to the first part of the paper, the constructed nonlinear price schedule turns out to be implementable for the case with two-dimensional private information if the weakened hazard rate condition employed in our discussion of concavity is satisfied. Since prices are time-independent, the policy is also immune to strategic buyer arrivals (which we do not model here explicitly). We also point out that a policy that varies with time but not with requested weight (whose asymptotic optimality in the complete information case was established by (Lin et al. 2008) is usually not optimal under incomplete information.

The paper is organized as follows. In Section 2 we present the dynamic model and the informational assumptions about values and weights. In Section 3 we characterize incentive compatible allocation policies. In Section 4 we focus on dynamic revenue maximization. We first characterize the revenue maximizing policy for the case where values are private information but weight requests are observable. We then offer two results that exhibit conditions under which the above policy is incentive compatible, and thus optimal also for the case where both values and weights are private information. Section 5 contains the asymptotic analysis.

2. The model

The designer has a "knapsack" of given capacity $C \in \mathbb{R}$ that he wants to allocate in a revenue maximizing way to several agents in at most $T < \infty$ periods. In each period, an impatient agent arrives with a demand for capacity characterized by a weight (or quantity request) w and by a per-unit value v.[3] While the realization of the random vector (w, v) is private information to the arriving agent, its distribution is assumed to be common knowledge and given by the joint cumulative distribution function $F(w, v)$, with continuously differentiable density $f(w, v) > 0$, defined on $[0, \infty)^2$. Demands are independent across different periods.[4]

In each period, the designer decides on a capacity to be allocated to the arriving agent (possibly none) and on a monetary payment. Type (w, v)'s utility is given by $wv - p$ if at price p he is allocated a capacity $w' \geq w$ and by $-p$ if he is assigned an insufficient capacity $w' < w$. Each agent observes the remaining capacity of the designer.[5] Finally, we assume that for all w, the conditional virtual value functions $\hat{v}(v, w) := v - (1 - F(v|w))/f(v|w)$ are unbounded as a function of v and strictly monotone increasing in v with $\partial\hat{v}(v, w)/\partial v > 0$ for all (w, v).

[3] It is an easy extension to assume that the arrival probability per period is given by $p < 1$.

[4] As pointed out by a referee, the results of Sections 3 and 4 apply also with the obvious modifications if types in different periods are independent, but not necessarily drawn from identical distributions.

[5] Alternatively, we can assume that each agent observes the entire history of the previous allocations. These assumptions are innocuous in the following sense: when we analyze revenue maximization in Section 4, we first solve for the optimal policy in the relaxed problem with observable weight types w. We then provide conditions for when this relaxed solution is implementable. Since in the case of observable weight requests, the seller cannot gain by hiding the available capacity, the seller cannot increase expected revenue by hiding the remaining capacity in the original problem.

3. Incentive compatible policies

To characterize the revenue maximizing scheme, we may restrict attention, without loss of generality, to direct mechanisms where every agent, upon arrival, reports a type (w, v) and the mechanism then specifies an allocation and a payment. In this section, we characterize incentive compatibility for a class of allocation policies that necessarily contains the revenue maximizing one. The schemes we develop also have an obvious and immediate interpretation as indirect mechanisms, where the designer sets menus of per-unit prices depending on time and on the remaining capacity.

An allocation rule is called *deterministic* and *Markovian* if, at any period $t = 1, \dots, T$ and for any possible type of agent arriving at t, it uses a nonrandom allocation rule that depends only on the arrival time t, on the declared type of the arriving agent, and on the still available capacity at period t, denoted by c. The restriction to these policies is innocuous as shown in Section 4.

We can assume without loss of generality that a deterministic Markovian allocation rule for time t with remaining capacity c has the form $\alpha_t^c : [0, +\infty)^2 \to \{1, 0\}$, where 1 (0) means that the reported capacity demand w is satisfied (not satisfied). Indeed, it never makes sense to allocate an insufficient quantity $0 < w' < w$ because individually rational agents are not willing to pay for this. Alternatively, allocating more capacity than the reported demand is useless as well: Such allocations do not further increase agents' utility while they may decrease continuation values for the designer. Let $q_t^c : [0, +\infty)^2 \to \mathbb{R}$ be the associated payment rule.

Proposition 1. *A deterministic, Markovian allocation rule $\{\alpha_t^c\}_{t,c}$ is implementable if and only if for every t and every c, it satisfies the following two conditions.*[6]

(i) For all (w, v), $v' \geq v$, $\alpha_t^c(w, v) = 1 \Rightarrow \alpha_t^c(w, v') = 1$.

(ii) The function $wp_t^c(w)$ is nondecreasing in w, where $p_t^c(w) = \inf\{v / \alpha_t^c(w, v) = 1\}$.[7]

When the above two conditions are satisfied, the allocation rule $\{\alpha_t^c\}_{t,c}$ together with the payment rule

$$q_t^c(w, v) = \begin{cases} wp_t^c(w) & \text{if } \alpha_t^c(w, v) = 1 \\ 0 & \text{if } \alpha_t^c(w, v) = 0 \end{cases}$$

constitute an incentive compatible policy.

See the Appendix for all proofs.

The threshold property embodied in condition (i) of the proposition is standard and is a natural feature of welfare maximizing rules under complete information. When there is incomplete information in the value dimension, this condition imposes limitations on the payments that can be extracted in equilibrium. Condition (ii) is new:

[6] Here we use "implementable" in the standard sense from the mechanism design literature. An allocation rule is implementable if we can associate to it a payment rule such that any agent finds it optimal to truthfully reveal her type when faced with the combined allocation–payment scheme.

[7] We set $p_t^c(w) = \infty$ if the set $\{v / \alpha_t^c(w, v) = 1\}$ is empty.

it reflects the limitations imposed in our model by the incomplete information in the weight dimension. We note that the above simple result is based on a combination of three main factors: (1) Due to our special utility function and to the pursued goal of revenue maximization, it is sufficient to consider only policies that allocate either the demanded weight to the agent or nothing. (2) The monotonicity requirement behind incentive compatibility boils down to the above simple conditions. (3) The integrability condition is automatically satisfied by all monotone allocation rules in the considered class. In general, one has to consider more allocation functions, more implications of monotonicity, and possibly an integrability constraint.

4. Dynamic revenue maximization

We first demonstrate how the dynamic revenue maximization problem can be solved if w is observable. This is, essentially, the dynamic programming problem analyzed by Papastavrou, Rajagopalan, and Kleywegt, translated from values to virtual values. Nevertheless, the logic of the derivation is somewhat involved, so we detail it below.

1. Without loss of generality, we can restrict attention to Markovian policies. The optimality of Markovian, possibly randomized, policies is standard for all models where, as is the case here, the per-period rewards and transition probabilities are history-independent; see, for example, Theorem 11.1.1 in Puterman (2005) which shows that for any history-dependent policy, there is a Markovian, possibly randomized, policy with the same payoff.

2. If there is incomplete information about v, but complete information about the weight requirement w, then Markovian, deterministic, and implementable policies are characterized for each t and c by the threshold property of condition (i) in Proposition 1.

3. Naturally, in the given revenue maximization problem with complete information about w, we need to restrict attention to interim individually rational policies where no agent ever pays more than the utility obtained from her actual capacity allocation. It is easy to see that for any Markov, deterministic, and implementable allocation rule α_t^c, the maximal, individually rational payment function that supports it is given by

$$q_t^c(w,v)=\begin{cases} wp_t^c(w) & \text{if } \alpha_t^c(w,v)=1\\ 0 & \text{if } \alpha_t^c(w,v)=0,\end{cases}$$

   where $p_t^c(w)=\inf\{v/\alpha_t^c(w,v)=1\}$ as defined in the above section. Otherwise, the designer pays some positive subsidy to the agent, and this cannot be revenue maximizing.

4. At each period t and for each remaining capacity c, the designer's problem under complete information about w is equivalent to a simpler, one-dimensional static problem where a known capacity needs to be allocated to the arriving agent and

where the seller has a salvage value for each remaining capacity: the salvage values in the static problem correspond to the continuation values in the dynamic version. Analogous to the analysis of Myerson (1981), each static revenue maximization problem has a monotone (in the sense of condition (i) in Proposition 1), nonrandomized solution as long as for any weight w, the agent's conditional virtual valuation $v - (1 - F(v|w))/f(v|w)$ is increasing in v.[8] If per-unit prices are set at $p_t^c(w)$ in period $t \leq T$ (so $T + 1 - t$ periods, including the current one, remain until the deadline) with remaining capacity c and if the optimal Markovian policy is followed from time $t + 1$ onward, the expected revenue $R(c, T + 1 - t)$ can be written as

$$\begin{aligned} R(c, T+1-t) = \int_0^c & w p_t^c(w)\bigl(1 - F(p_t^c(w)|w)\bigr)\bar{f}_w(w)\,dw \\ & + \int_0^c \bigl[\bigl(1 - F(p_t^c(w)|w)\bigr)R^*(c - w, T - t) \\ & + F(p_t^c(w)|w)R^*(c, T - t)\bigr]\bar{f}_w(w)\,dw, \end{aligned}$$

where $\bar{f}_w$ denotes the marginal density in w and where R^* denotes optimal revenues, with $R^*(c, 0) = 0$ for all c. The first-order conditions for the revenue maximizing unit prices $p_t^c(w)$ are given by[9]

$$w\left(p_t^c(w) - \frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) = R^*(c, T - t) - R^*(c - w, T - t).$$

5. By backward induction, and by the above reasoning, the seller has a Markov, nonrandomized optimal policy in the dynamic problem with complete information about w. Note also that, by a simple duplication argument, $R^*(c, T + 1 - t)$ must be monotone nondecreasing in c.

Points 1, 4, and 5 above imply that the restriction to deterministic and Markovian allocation policies is without loss of generality. If the above solution to the relaxed problem satisfies the incentive compatibility constraint in the weight dimension, i.e., if $wp_t^c(w)$ happens to be monotone as required by condition (ii) of Proposition 1, then the associated allocation where $\alpha_t^c(w, v) = 1$ if and only if $v \geq p_t^c(w)$ is also implementable in the original problem with incomplete information about both v and w. It then constitutes the revenue maximizing scheme that we are after. The next example illustrates that condition (ii) of Proposition 1 can be binding.

[8]Note that the optimal policy continues to be deterministic even if virtual valuations are not monotonic. This follows by a similar argument to the one given by Riley and Zeckhauser (1983). We nevertheless keep the monotonicity assumption for simplicity and because we need related conditions for some of the subsequent results.

[9]By our assumption of unbounded conditional virtual values (which is a mild assumption on distributions with unbounded support), these first-order conditions always admit a solution and must, therefore, be satisfied at the optimum.

Example 1. Assume that $T = 1$. The distribution of the agents' types is given by the following stochastic process. First, the weight request w is realized according to an exponential distribution with parameter λ. Next, the per-unit value of the agent is sampled from the distribution

$$F(v|w) = \begin{cases} 1 - e^{-\overline{\lambda} v} & \text{if } w > w^* \\ 1 - e^{-\underline{\lambda} v} & \text{if } w \le w^*, \end{cases}$$

where $\overline{\lambda} > \underline{\lambda}$ and $w^* \in (0, c)$.

In this case, for an observable weight request, the seller charges the take-it-or-leave-it offer of $1/\underline{\lambda}$ $(1/\overline{\lambda})$ per unit if the weight request is smaller (larger) than or equal to w^*. This implies that

$$wp_t^c(w) = \begin{cases} \dfrac{w}{\overline{\lambda}} & \text{if } w > w^* \\ \dfrac{w}{\underline{\lambda}} & \text{if } w \le w^* \end{cases}$$

and, therefore, $wp_t^c(w)$ is not monotone. ◊

4.1 *The hazard rate stochastic ordering*

A simple sufficient condition that guarantees implementability of the relaxed solution is a particular stochastic ordering of the conditional distributions of per-unit values: the conditional distribution given a higher weight should be (weakly) statistically higher in the hazard rate order than the conditional distribution given a lower weight. This is similar to conditions found in static frameworks by Pai and Vohra, Iyengar and Kumar, or Blackorby and Szalay.

Theorem 1. *For each c, t, and w, let $p_t^c(w)$, denote the solution to the revenue maximizing problem under complete information about w, determined recursively by the Bellman equation*

$$w\left(p_t^c(w) - \frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) = R^*(c, T-t) - R^*(c-w, T-t). \tag{1}$$

Assume that the following conditions hold.

(i) For any w, the conditional hazard rate $f(v|w)/(1 - F(v|w))$ is nondecreasing in v.[10]

(iii) For any $w' \ge w$, and for any v, $f(v|w)/(1 - F(v|w)) \ge f(v|w')/(1 - F(v|w'))$.

Then $wp_t^c(w)$ is nondecreasing in w and, consequently, the underlying allocation where $\alpha_t^c(w, v) = 1$ if and only if $v \ge p_t^c(w)$ is implementable. In particular, (1) characterizes the revenue maximizing scheme under incomplete information about both values and weights.

[10]Note that this condition already implies the needed monotonicity in v of the conditional virtual value for all w.

An important special case for which the conditions of Theorem 1 hold is where the distribution of per-unit values is independent of the distribution of weights and has an increasing hazard rate.

4.2 *The role of concavity*

A major result for the case where capacity comes in discrete units and where all weights are equal is that optimal expected revenue is concave in capacity (see Gallego and van Ryzin 1994 for a continuous time framework with Poisson arrivals and Bitran and Mondschein 1997 for a discrete time setting). This is a very intuitive property since it says that additional capacity is more valuable to the designer when capacity itself is scarce. Due to the more complicated combinatorial nature of the knapsack problem with heterogeneous weights, concavity need not generally hold (see Papastavrou et al. 1996 for examples where concavity of expected welfare in the framework with complete information fails). When concavity does hold, the optimal per-unit virtual value thresholds for the relaxed problem increase with weight, which facilitates implementation for the case of two-dimensional private information.

Our main result in this subsection identifies a condition on the distribution of types that, together with concavity of the expected revenue in the remaining capacity, ensures that for each t and c, $wp_t^c(w)$ is increasing.

Theorem 2. *Assume the following conditions.*

(i) The expected revenue $R^*(c, T+1-t)$ is a concave function of c for all times t.

(ii) For any $w \leq w'$, $v - (1-F(v|w))/f(v|w) \geq vw/w' - (1-F(vw/w'|w'))/f(vw/w'|w')$.

For each c, t, and w, let $p_t^c(w)$ denote the solution to the revenue maximizing problem under complete information about w, determined recursively by (1). Then $wp_t^c(w)$ is nondecreasing in w, and hence the underlying allocation where $\alpha_t^c(w, v) = 1$ if and only if $v \geq p_t^c(w)$ is implementable. In particular, (1) characterizes the revenue maximizing scheme under incomplete information about both values and weights.

Remark 1. The sufficient conditions for implementability used in Theorem 1 are, taken together, stronger than condition (ii) in Theorem 2. To see this, assume that for any w, the conditional hazard rate $f(v|w)/(1-F(v|w))$ is increasing in v, and that for any $w' \geq w$ and for all v, $f(v|w)/(1-F(v|w)) \geq f(v|w')/(1-F(v|w'))$. This yields:

$$v - \frac{1-F(v|w)}{f(v|w)} \geq \frac{vw}{w'} - \frac{1-F\left(\frac{vw}{w'}|w\right)}{f\left(\frac{vw}{w'}|w\right)} \geq \frac{vw}{w'} - \frac{1-F\left(\frac{vw}{w'}|w'\right)}{f\left(\frac{vw}{w'}|w'\right)},$$

where the first inequality follows by the monotonicity of the hazard rate, and the second follows by the stochastic order assumption. Note also that condition (ii) of Theorem 2 can be formulated as requiring that the functions $\alpha v - (1-F(\alpha v|w/\alpha))/f(\alpha v|w/\alpha)$ are nondecreasing in α. Finally, note that this condition will play an important role for implementability of the asymptotically optimal policy that we construct in Section 5 below.

We next modify a result of Papastavrou et al. (1996) so as to identify conditions on the joint distribution $F(w,v)$ that imply concavity of expected revenue with respect to c for all periods, as required by the Theorem 2.[11] It is convenient to introduce the joint distribution of weight and total valuation $u = vw$, which we denote by $G(w,u)$ with density $g(w,u)$. By means of a transformation of variables, the densities f and g are related by $wg(w,wv) = f(w,v)$. In particular, marginal densities in w coincide, i.e.,

$$\bar{f}_w(w) = \int_0^\infty f(w,v)\,dv = \int_0^\infty g(w,u)\,du = \bar{g}_w(w).$$

An increasing virtual value implies that the virtual total value is increasing in u with strictly positive derivative for any given w:

$$\hat{u}(u,w) := u - \frac{1-G(u|w)}{g(u|w)} = wv - \frac{1-F(v|w)}{f(v|w)/w} = w\hat{v}(v,w).$$

We write $\hat{u}^{-1}(\hat{u},w)$ for the inverse of $\hat{u}(u,w)$ with respect to u and define a distribution $\hat{G}(\hat{u},w)$ by both $\hat{G}(\hat{u}|w) := G(\hat{u}^{-1}(\hat{u},w)|w)$ for all w and $\hat{\bar{g}}_w(w) := \bar{g}_w(w)$. On the level of $\hat{v}$, this corresponds to $\hat{F}(\hat{v}|w) = F(\hat{v}^{-1}(\hat{v},w)|w)$ and $\hat{\bar{f}}_w(w) = \bar{f}_w(w)$.

Theorem 3. *Assume that the conditional distribution $\hat{G}(w|\hat{u})$ is concave in w for all $\hat{u}$, that both $\hat{g}(w|\hat{u})$ and $d\hat{g}(w|\hat{u})/dw$ are bounded, and that the total virtual value $\hat{u}$ has a finite mean. Then, in the revenue maximization problem where the designer has complete information about w, the expected revenue $R^*(c, T+1-t)$ is concave as a function of c for all times t.*

Example 2. A simple setting where the conditions of Theorem 2 are satisfied while those of Theorem 1 are violated is obtained as follows. Assume that $G(w,u)$ is such that u and w are independent, the hazard rate $g_u(u)/(1-G_u(u))$ is nondecreasing, and G_w is concave.[12] Then condition (i) of Theorem 2 is satisfied according to Theorem 3 because the $\hat{G}(w|\hat{u})$ are concave. Consider then $w < w'$. By independence of u and w, we have $w'\hat{v}(vw/w', w') = \hat{u}(vw,w') = \hat{u}(vw,w) = w\hat{v}(v,w)$ and hence $\hat{v}(v,w) = (w'/w)\hat{v}(vw/w',w')$. As $w'/w > 1$, this implies condition (ii) of Theorem 2 in the relevant domain where virtual values are nonnegative. However, as we show now, condition (ii) of Theorem 1, i.e., the hazard rate ordering, is violated. Indeed, the equation we have just derived implies also that $f(v|w)/(1-F(v|w)) = (w/w')f(vw/w'|w')/(1-F(vw/w'|w'))$. But the conditional hazard rates of F are nondecreasing (because G_u has nondecreasing hazard rate) and $w/w' < 1$, so that $f(v|w)/(1-F(v|w)) = (w/w')f(vw/w'|w')/(1-F(vw/w'|w')) < f(v|w')/(1-F(v|w'))$, which contradicts the hazard rate ordering of Theorem 1. $\Diamond$

[11] In the Appendix we also provide an elementary proof of the result of Papastavrou et al. (1996), since a proof is neither contained in the above-mentioned paper nor in the related one by Kleywegt and Papastavrou (2001). Moreover, we were unable to find a more general result from finite horizon stochastic dynamic programming that ensures concavity of expected value in the state variable c, which is only a part of the relevant state description.

[12] We also assume that the other mild technical conditions of Theorem 3 are satisfied.

5. Asymptotically optimal and time-independent pricing

The optimal policy identified above requires price adjustments in every period and for any quantity request w. These dynamics are arguably too complicated to be applied in practice. Gallego and van Ryzin (1994) use an asymptotic argument to show that the theoretical gain from optimal dynamic pricing compared to a suitably chosen, time-independent policy is usually small in the setting with unit demands. Our main theorem in this section extends their result to the dynamic knapsack problem with general distribution of types. We construct a static nonlinear price schedule that uses the existing correlations between w and v, and show that it is asymptotically optimal if both capacity and time horizon go to infinity.

While the basic strategy of the proof follows the suggestion made by Gallego and van Ryzin, there are several major differences. In fact, in Section 5 of their paper these authors also consider the case of heterogeneous capacity demands. However, they assume that weights and values are independent and, most importantly, their optimality benchmark does not even allow per-unit prices to depend on weight requests. But, as we saw above, such weight dependency is a general property of the dynamically optimal solution, even if w and v are independent. We therefore take our solution of the relaxed problem as the optimality benchmark and we also consider general type distributions F.

As above, we start by focusing on the case of observable weights. We then show that condition (ii) of Theorem 2 is a sufficient condition for implementability for the case with two-dimensional private information.

Like Gallego and van Ryzin, we first solve a simpler, suitably chosen deterministic maximization problem. The revenue obtained in the solution to that problem provides an upper bound for the optimal expected revenue of the stochastic problem, and the solution itself suggests the use of per-unit prices that depend on weight requests, but that are constant in time. We next show that the derived policy is asymptotically optimal also in the original stochastic problem where both capacity and time go to infinity: the ratio of expected revenue from following the considered policy over expected revenue from the optimal Markovian policy converges to 1. Moreover, there are various ways to quantify this ratio for moderately large capacities and time horizons.

Let us first recall some assumptions and introduce further notation. The marginal density $\bar{f}_w(w)$ and the conditional densities $f(v|w)$ pin down the distribution of (independent) arriving types $(w_t, v_t)_{t=1}^T$. Given w, the demanded per-unit price p and the probability λ^w of a request being accepted are related by $\lambda^w(p) = 1 - F(p|w)$. Let $p^w(\lambda)$ be the inverse of λ and note that this is well defined on $(0, 1]$. Because of monotonicity of conditional virtual values, the instantaneous (expected) per-unit revenue functions $r^w(\lambda) := \lambda p^w(\lambda)$ are strictly concave and each one attains a unique interior maximum. Indeed, $p^w(\lambda) = F(\cdot|w)^{-1}(1-\lambda)$ and hence

$$\frac{d}{d\lambda} r^w(\lambda) = p^w(\lambda) - \lambda \frac{1}{f(p^w(\lambda)|w)} = p^w(\lambda) - \frac{1 - F(p^w(\lambda)|w)}{f(p^w(\lambda)|w)} = \hat{v}(p^w(\lambda), w)$$

$$\frac{d^2}{d\lambda^2} r^w(\lambda) = -\left(\frac{\partial}{\partial v}\hat{v}\right)(p^w(\lambda), w) \frac{1}{f(p^w(\lambda)|w)} < 0.$$

Consequently, r^w is strictly concave, strictly increasing up to the $\lambda^{w,*}$ that satisfies $\hat{v}(p^w(\lambda^{w,*}), w) = 0$, and strictly decreasing from there on.

5.1 *The deterministic problem*

We now formulate an auxiliary deterministic problem that closely resembles the relaxed stochastic problem. Let $\mathrm{Cap}\colon (0, \infty) \to (0, \infty)$, $w \mapsto \mathrm{Cap}(w)$ be a measurable function. Consider the problem:

$$\max_{\mathrm{Cap}(\cdot)} \int_0^\infty \max_{(\lambda_t^w)_{t=1,\dots,T}} \left(\sum_{t=1}^T r^w(\lambda_t^w) \right) w \bar{f}_w(w)\, dw \tag{2}$$

subject to

$$\sum_{t=1}^T \lambda_t^w w \bar{f}_w(w) \leq \mathrm{Cap}(w) \text{ a.s.} \quad \text{and} \quad \int_0^\infty \mathrm{Cap}(w)\, dw \leq C. \tag{3}$$

In words, we analyze a problem where the following statements are true.

1. The capacity C needs to be divided into capacities $\mathrm{Cap}(w)$, one for each w.
2. In each w subproblem, a deterministic quantity request of $w\bar{f}_w(w)$ arrives in each period, and λ_t^w determines a share (not a probability!) of this request that is accepted and sold at per-unit price $p^w(\lambda_t^w)$.
3. In each subproblem, the allocated capacity over time cannot exceed $\mathrm{Cap}(w)$ and total allocated capacity in all subproblems $\int_0^\infty \mathrm{Cap}(w)\, dw$ cannot exceed C.
4. The designer's goal is to maximize total revenue. We call the revenue at the solution $R^d(C, T)$.

As r^w is strictly concave and increasing up to $\lambda^{w,*}$, it is straightforward to verify that, given a choice $\mathrm{Cap}(w)$, the solution to the w subproblem

$$\max_{(\lambda_t^w)_{t=1,\dots,T}} \left(\sum_{t=1}^T r^w(\lambda_t^w) \right) w \bar{f}_w(w) \quad \text{such that} \quad \sum_{t=1}^T \lambda_t^w w \bar{f}_w(w) \leq \mathrm{Cap}(w)$$

is given by

$$\lambda_t^w \equiv \lambda^{w,d} := \begin{cases} \lambda^{w,*} & \text{if } \lambda^{w,*} \leq \dfrac{\mathrm{Cap}(w)}{Tw\bar{f}_w(w)} \\ \dfrac{\mathrm{Cap}(w)}{Tw\bar{f}_w(w)} & \text{else.} \end{cases} \tag{4}$$

Accordingly, the revenue in the w subproblem is $r^w(\lambda^{w,d}) Tw\bar{f}_w(w)$.

Proposition 2. *The solution to the deterministic problem given by* (2) *and* (3) *is characterized by* *(i)* $\hat{v}(p^w(\lambda^{w,d}), w) = \beta(C, T) = \text{const}$

(ii) $\lambda_t^w = \lambda^{w,d} = \frac{\text{Cap}(w)}{Tw\bar{f}_w(w)}$

(iii) $\int_0^\infty \text{Cap}(w)\, dw = \min(C, T\int_0^\infty \lambda^{w,*} w\bar{f}_w(w)\, dw)$.

To get an intuition for the above result, observe that the marginal increase of the optimal revenue for the w subproblem from marginally increasing $\text{Cap}(w)$ is

$$\left(\frac{d}{d\lambda} r^w\right)\left(\frac{\text{Cap}(w)}{Tw\bar{f}_w(w)}\right) = \hat{v}(p^w(\lambda^{w,d}), w) \quad \text{if } \lambda^{w,*} > \frac{\text{Cap}(w)}{Tw\bar{f}_w(w)}$$

and 0 otherwise.

Proposition 2 says that, optimally, the capacity should be split in such a way that the marginal revenue from increasing $\text{Cap}(w)$ is the same for all w. Actually solving the problem amounts to the simple static exercise of determining the constant $\beta(C, T)$ in accordance with the integral feasibility constraint.

The above construction is justified by the following two-step argument: on the one hand, we show in Theorem 4 below that the optimal revenue in the deterministic problem, $R^d(C, T)$, bounds from above the optimal revenue in the original stochastic case. On the other hand, as we show in Section 5.2, the optimal solution of the deterministic problem serves to define a simple time-independent policy that in the stochastic problem captures revenues $R^{\text{TI}}(C, T)$ such that $R^{\text{TI}}(C, T)/R^d(C, T)$ converges to 1 as C and T go to infinity. Combining these two points yields the kind of asymptotic optimality result we want to establish.

Since we assume here that weights are observable, a Markovian policy α for the original stochastic problem is characterized by the acceptance probabilities $\lambda_t^{w_t}[c_t]$ contingent on current time t, remaining capacity c_t, and weight request w_t. Expected revenue from policy α at the beginning of period t (i.e., when there are $(T - t + 1)$ periods left) with remaining capacity c_t is given by

$$R_\alpha(c_t, T - t + 1) = E_\alpha\left[\sum_{s=t}^{T} w_s p^{w_s}(\lambda_s^{w_s}[c_s]) I_{\{v_s \geq p^{w_s}(\lambda_s^{w_s}[c_s])\}}\right]$$

such that

$$\sum_{s=t}^{T} w_s I_{\{v_s \geq p^{w_s}(\lambda_s^{w_s}[c_s])\}} \leq c_t.$$

Here, the constraint must hold almost surely when following α. As before, we write $R^*(c_t, T - t + 1)$ for the optimal revenue, i.e., the supremum of expected revenues taken over all feasible Markovian policies α.

Theorem 4. *For any capacity C and deadline T, it holds that $R^*(C, T) \leq R^d(C, T)$.*

5.2 *A simple policy for the stochastic problem*

Having established the upper bound of Theorem 4, we now proceed with the second part of our two-step argument outlined in the preceding section. We use the optimal solution of the deterministic problem to define a w-contingent yet time-independent policy α_{TI} for the stochastic case as follows.

1. Given C and T, solve the deterministic problem to obtain $\beta(C,T)$, $\lambda^{w,d}$, and thus $p^{w,d} := p^w(\lambda^{w,d}) = \hat{v}^{-1}(\beta(C,T), w)$.
2. In the stochastic problem charge these weight-contingent prices $p^{w,d}$ for the entire time horizon, provided that the quantity request does not exceed the remaining capacity. Otherwise, charge a price equal to $+\infty$ (i.e., reject the request).

Note that under condition (ii) of Theorem 2, the time-independent policy α_{TI} *is also implementable if weights are not observable!* Indeed, setting all virtual valuation thresholds equal to a constant is like setting them optimally for linear and hence concave salvage values.

We now determine how well the time-independent policy constructed above performs compared to the optimal Markovian policy. Recall that we do this by comparing its expected revenue, $R^{\mathrm{TI}}(C,T)$, with the optimal revenue in the deterministic problem, $R^d(C,T)$, which, as we know by Theorem 4, provides an upper bound for the optimal revenue in the stochastic problem, $R^*(C,T)$.

Theorem 5. *(i) For any joint distribution of values and weights,*

$$\lim_{C,T\to\infty, C/T=\text{const}} \frac{R^{\mathrm{TI}}(C,T)}{R^d(C,T)} = 1.$$

(ii) Assume that w and v are independent. Then

$$\frac{R^{\mathrm{TI}}(C,T)}{R^d(C,T)} \geq 1 - \frac{\sqrt{E[w^2]/E[w]}}{2\sqrt{\min(C, \lambda^* E[w]T)}}.$$

In particular, $\lim_{\min(C,T)\to\infty} \frac{R^{\mathrm{TI}}(C,T)}{R^d(C,T)} = 1$.

We have chosen to focus on these two general limit results, but various other quantitative results could be proven by similar techniques at the expense of slightly more technical effort and possibly some further assumptions on the distribution F.[13] This should be clear from the proof in the Appendix. Note that since policy α_{TI} is stationary, it does not generate incentives to postpone arrivals even in a more complex model where buyers are patient and can choose their arrival time.

[13]Since $R^d(C,T) \geq R^*(C,T) \geq R^{\mathrm{TI}}(C,T)$ (the first inequality is Theorem 4 and the second follows by optimality), our estimate in Theorem 5(ii) immediately extends to $R^*(C,T)/R^d(C,T)$ or to $R^{\mathrm{TI}}(C,T)/R^*(C,T)$.

Remark 2. In a complete information knapsack model, Lin et al. (2008) study policies that start by accepting only high value requests and then switch over to accepting also lower values as time goes by. They establish asymptotic optimality of such policies (with carefully chosen switchover times) as available capacity and time go to infinity. In other words, their prices are time-dependent but do not condition on the weight request. It is easy to show that, in our incomplete information model, such policies are, in general, suboptimal. Consider first a one-period example where the seller has capacity 2 and where the arriving agent has either a weight request of 1 or 2 (equally likely). If the weight request is 1 (2), the agent's per-unit value distributes uniformly between 0 and 1 (between 1 and 2). The optimal mechanism in this case is as follows: if the buyer requests one unit, the seller sells it for a price of 0.5, and if the buyer requests two units, the seller sells each unit at a price of 1. Note that this policy is implementable since the requested per-unit price is monotonically increasing in the weight request. The expected revenue is 9/8. If, however, the seller is forced to sell all units at the same per-unit price without conditioning on the weight request, he will charge the price of 1 for each unit, yielding an expected revenue of 1, and thus lose 1/8 versus the optimal policy. Now replicate this problem so that there are T periods and capacity $C=2T$. Then the expected revenue from the optimal mechanism is $9/8T$, while the expected revenue from the constrained mechanism is only T. Obviously, the constrained mechanism is not asymptotically optimal.

Appendix

Proof of Proposition 1. $\Longrightarrow$. So assume that conditions (i) and (ii) are satisfied, and define for any t, c,

$$q_t^c(w,v)=\begin{cases} wp_t^c(w) & \text{if } \alpha_t^c(w,v)=1\\ 0 & \text{if } \alpha_t^c(w,v)=0.\end{cases}$$

Consider then an arrival of type (w,v) in period t with remaining capacity c. There are two cases.

(a) $\alpha_t^c(w,v)=1$. In particular, $v\geq p_t^c(w)$. Then truth-telling yields utility $w(v-p_t^c(w))\geq 0$. Assume that the agent reports instead $(\widehat{w},\widehat{v})$. If $\alpha_t^c(\widehat{w},\widehat{v})=0$, then the agent's utility is zero and the deviation is not profitable. Assume then that $\alpha_t^c(\widehat{w},\widehat{v})=1$. By the form of the utility function, a report of $\widehat{w}<w$ is never profitable. But for $\widehat{w}\geq w$, the agent's utility is $wv-\widehat{w}p_t^c(\widehat{w})\leq w(v-p_t^c(w))$, where we used condition (ii). Therefore, such a deviation is also not profitable.

(b) $\alpha_t^c(w,v)=0$. In particular, $v\leq p_t^c(w)$. Truth-telling yields here utility of zero. Assume that the agent reports instead $(\widehat{w},\widehat{v})$. If $\alpha_t^c(\widehat{w},\widehat{v})=0$, then the agent's utility remains zero and the deviation is not profitable. Assume then that $\alpha_t^c(\widehat{w},\widehat{v})=1$. By the form of the utility function, a report of $\widehat{w}<w$ is never profitable. Thus, consider the case where $\widehat{w}\geq w$. In this case, the agent's utility is $wv-\widehat{w}p_t^c(\widehat{w})\leq w(v-p_t^c(w))\leq 0$, where we used condition (ii). Therefore, such a deviation is also not profitable.

$\Longleftarrow$. Consider now an implementable, deterministic, and Markovian allocation policy $\{\alpha_t^c\}_{t,c}$. Assume first, by contradiction, that condition (i) in the statement of the proposition is not satisfied. Then there exist (w, v) and (w, v') such that $v' > v$, $\alpha_t^c(w, v) = 1$, and $\alpha_t^c(w, v') = 0$. We obtain the chain of inequalities $wv' - q_t^c(w, v) > wv - q_t^c(w, v) \geq -q_t^c(w, v')$, where the second inequality follows by incentive compatibility for type (w, v). This shows that a deviation to a report (w, v) is profitable for type (w, v'), a contradiction to implementability. Therefore, condition (i) must hold.

In particular, note that for any two types who have the same weight request, (w, v) and (w, v'), if both are accepted, i.e., $\alpha_t^c(w, v) = \alpha_t^c(w, v') = 1$, the payment must be the same (otherwise the type that needs to make the higher payment would deviate and report the other type). Denote this payment by $r_t^c(w)$. Note also that any two types (w, v) and (w', v') such that $\alpha_t^c(w, v) = \alpha_t^c(w', v') = 0$ must also make the same payment (otherwise the type that needs to make the higher payment would deviate and report the other type) and denote this payment by s.

Assume now, by contradiction, that condition (ii) does not hold. Then there exist w and w' such that $w' > w$ but $w' p_t^c(w') < wp_t^c(w)$. In particular, $w' p_t^c(w') < \infty$ and therefore $p_t^c(w') < \infty$.

Assume first that $p_t^c(w) < \infty$. We have $w' p_t^c(w') - r_t^c(w') = wp_t^c(w) - r_t^c(w) = -s$ because, by incentive compatibility, both types $(w,\, p_t^c(w))$ and $(w',\, p_t^c(w'))$ must be indifferent between getting their request and not getting it. Since by assumption $w' p_t^c(w') < wp_t^c(w)$, we obtain that $r_t^c(w') < r_t^c(w)$. Consider now a type (w, v) for which $v > p_t^c(w)$. By reporting truthfully, this type gets utility $wv - r_t^c(w)$, while by deviating to (w', v), he gets utility $wv - r_t^c(w') > wv - r_t^c(w)$, a contradiction to incentive compatibility.

Assume now that $p_t^c(w)$ is infinite, and therefore $wp_t^c(w)$ is infinite. Consider a type (w', v) where $v > p_t^c(w')$. The utility of this type is $w'v - r_t^c(w') > w' p_t^c(w') - r_t^c(w') = -s$. In particular, $r_t^c(w')$ must be finite. By reporting truthfully, a type (w, v) gets utility $-s$, while by deviating to a report of (w', v), he gets $wv - r_t^c(w')$. For v large enough, we obtain $wv - r_t^c(w') > -s$, a contradiction to implementability.

Thus, condition (ii) must hold and, in particular, the payment $r_t^c(w)$ is monotonic in w. $\square$

Proof of Theorem 1. Let $w < w'$. We need to show that $wp_t^c(w) - w' p_t^c(w') \leq 0$. If $p_t^c(w) \leq p_t^c(w')$, the result is clear. Assume then that $p_t^c(w) > p_t^c(w')$. We obtain the chain of inequalities

$$\begin{aligned} & w\left(\frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) - w'\left(\frac{1 - F(p_t^c(w')|w')}{f(p_t^c(w')|w')}\right) \\ & \qquad \leq w'\left(\frac{1 - F(p_t^c(w)|w)}{f(p_t^c(w)|w)} - \frac{1 - F(p_t^c(w')|w')}{f(p_t^c(w')|w')}\right) \\ & \qquad \leq w'\left(\frac{1 - F(p_t^c(w')|w)}{f(p_t^c(w')|w)} - \frac{1 - F(p_t^c(w')|w')}{f(p_t^c(w')|w')}\right) \leq 0, \end{aligned}$$

where the second inequality follows by the monotonicity of the hazard rate and the third follows by the hazard rate ordering condition.

Since $R^*(c-w, T-t)$ is monotonically decreasing in w, we obtain that

$$w\left(p_t^c(w) - \frac{1-F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) \le w'\left(p_t^c(w') - \frac{1-F(p_t^c(w')|w')}{f(p_t^c(w')|w')}\right) \quad \Leftrightarrow$$

$$wp_t^c(w) - w' p_t^c(w') \le w\left(\frac{1-F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) - w'\left(\frac{1-F(p_t^c(w')|w')}{f(p_t^c(w')|w')}\right) \le 0,$$

where the last inequality follows by the derivation above. Hence $wp_t^c(w) - w' p_t^c(w') \le 0$ as desired. $\square$

Proof of Theorem 2. For any concave function ϕ and for any $x < y < z$ in its domain, the well known "Three Chord Lemma" asserts that

$$\frac{\phi(y)-\phi(x)}{y-x} \ge \frac{\phi(z)-\phi(x)}{z-x} \ge \frac{\phi(z)-\phi(y)}{z-y}.$$

Then consider $w < w'$ and let $x = c - w' < y = c - w < z = c$. For the case of a concave revenue, the lemma then yields

$$\begin{aligned}\frac{R^*(c-w, T-t) - R^*(c-w', T-t)}{w'-w} &\ge \frac{R^*(c, T-t) - R^*(c-w', T-t)}{w'} \\ &\ge \frac{R^*(c, T-t) - R^*(c-w, T-t)}{w}.\end{aligned}$$

We obtain, in particular,

$$\begin{aligned}p_t^c(w') - \frac{1-F(p_t^c(w')|w')}{f(p_t^c(w')|w')} &= \frac{R^*(c, T-t) - R^*(c-w', T-t)}{w'} \\ &\ge \frac{R^*(c, T-t) - R^*(c-w, T-t)}{w} \\ &= p_t^c(w) - \frac{1-F(p_t^c(w)|w)}{f(p_t^c(w)|w)},\end{aligned}$$

which yields

$$p_t^c(w') - \frac{1-F(p_t^c(w')|w')}{f(p_t^c(w')|w')} \ge p_t^c(w) - \frac{1-F(p_t^c(w)|w)}{f(p_t^c(w)|w)} \ge \frac{w}{w'} p_t^c(w) - \frac{1-F\left(\frac{w}{w'} p_t^c(w)|w'\right)}{f\left(\frac{w}{w'} p_t^c(w)|w'\right)},$$

where the last inequality follows by the condition in the statement of the Theorem. Since virtual values are increasing, this yields $p_t^c(w') \ge (w/w') p_t^c(w) \Leftrightarrow w' p_t^c(w') \ge wp_t^c(w)$ as desired. $\square$

For the proof of Theorem 3, we first need a lemma on maximization of expected welfare under complete information. The result appears (without proof) in Papastavrou et al. (1996).

Lemma 1. *Assume that the total value u has finite mean, and that both $g(w|u)$ and $dg(w|u)/dw$ are bounded and continuous. Consider the allocation policy that maximizes expected welfare under complete information (i.e., upon arrival the agent's type is re-*

vealed to the designer). If $G(w|u)$ is concave in w for all u, then the optimal expected welfare, denoted U_t^c, is twice continuously differentiable and concave in the remaining capacity c for all periods $t \leq T$.

Proof. Note that, for notational convenience throughout this proof, we index optimal expected welfare by the current time t and not by periods remaining to deadline. By standard arguments, the optimal policy for this unconstrained dynamic optimization problem is deterministic and Markovian, and U_t^c is nondecreasing in remaining capacity c by a simple strategy duplication argument. Moreover, the optimal policy can be characterized by weight thresholds $w_t^c(u) \leq c$: If c remains at time t and a request whose acceptance would generate value u arrives, then it is accepted if and only if $w \leq w_t^c(u)$. If $U_{t+1}^c \geq u$, then the weight threshold must satisfy the indifference condition

$$u = U_{t+1}^c - U_{t+1}^{c-w_t^c(u)}. \tag{5}$$

Otherwise, we have $w_t^c(u) = c$.

We now prove the lemma by backward induction. At time $t = T$, i.e., in the deadline period, it holds that

$$U_T^c = \int_0^\infty G(c|u) u \bar{g}_u(u)\, du.$$

This is concave in c because all $G(c|u)$ are concave by assumption, because $u\bar{g}_u(u)$ is positive, and because the distribution of u has a finite mean. Since both $g(w|u)$ and $dg(w|u)/dw$ are bounded and continuous, U_T^c is also twice continuously differentiable.

Suppose now that the lemma has been proven down to time $t+1$. The optimal expected welfare at t provided that capacity c remains may be written as

$$U_t^c = \int_0^\infty \left[uG(w_t^c(u)|u) + \int_0^{w_t^c(u)} U_{t+1}^{c-w} g(w|u)\, dw + \bigl(1 - G(w_t^c(u)|u)\bigr) U_{t+1}^c \right] \bar{g}_u(u)\, du. \tag{6}$$

We proceed to show concavity with respect to c of the term in brackets, for all u. This in turn implies concavity of U_t^c and hence, with a short additional argument for differentiability, is sufficient to conclude the induction step. We distinguish the cases $u > U_{t+1}^c$ for which the indifference condition (5) does not hold and $u \leq U_{t+1}^c$ for which it does. For both cases, we demonstrate that the second derivative (one-sided if necessary) of the bracket term with respect to c is nonpositive and thus we establish global concavity.

Case 1. $u > U_{t+1}^c$. The bracket term becomes $uG(c|u) + \int_0^c U_{t+1}^{c-w} g(w|u)\, dw + (1 - G(c|u))U_{t+1}^c$. By continuity of U_{t+1}^c, this representation also holds in a small interval around c. We find

$$\begin{aligned} \frac{d}{dc}\left[uG(c|u) + \int_0^c U_{t+1}^{c-w} g(w|u)\, dw + (1 - G(c|u))U_{t+1}^c \right] \\ = ug(c|u) + \int_0^c \frac{d}{dc} U_{t+1}^{c-w} g(w|u)\, dw + U_{t+1}^0 g(c|u) \\ - g(c|u)U_{t+1}^c + (1 - G(c|u)) \frac{d}{dc} U_{t+1}^c \end{aligned}$$

$$= (u - U_{t+1}^c)g(c|u) + \int_0^c \frac{d}{dc} U_{t+1}^{c-w} g(w|u)\, dw + (1 - G(c|u)) \frac{d}{dc} U_{t+1}^c$$

and

$$\begin{aligned}
&\frac{d^2}{dc^2}\left[uG(c|u) + \int_0^c U_{t+1}^{c-w} g(w|u)\, dw + (1 - G(c|u)) U_{t+1}^c \right] \\
&\quad = (u - U_{t+1}^c) g'(c|u) - g(c|u) \frac{d}{dc} U_{t+1}^c + \int_0^c \frac{d^2}{dc^2} U_{t+1}^{c-w} g(w|u)\, dw \qquad (7) \\
&\qquad + \frac{d}{dw} U_{t+1}^w \Big|_{w=0} g(c|u) - g(c|u) \frac{d}{dc} U_{t+1}^c + (1 - G(c|u)) \frac{d^2}{dc^2} U_{t+1}^c.
\end{aligned}$$

The last term is nonpositive by the concavity of U_{t+1}^c; the first term is nonpositive because $u > U_{t+1}^c$ and because $G(c|u)$ has a nonincreasing density by assumption. In addition, $g(c|u)(dU_{t+1}^c/dc)$ is nonnegative and hence (7) is bounded from above by

$$\int_0^c \frac{d^2}{dc^2} U_{t+1}^{c-w} g(w|u)\, dw + g(c|u) \left(\frac{d}{dw} U_{t+1}^w \Big|_{w=0} - \frac{d}{dc} U_{t+1}^c \right).$$

But $\int_0^c (d^2 U_{t+1}^{c-w}/dc^2) g(w|u)\, dw$ may be bounded from above by $g(c|u) \times \int_0^c (d^2 U_{t+1}^{c-w}/dc^2)\, dw$ because of the decreasing density and because $d^2 U_{t+1}^{c-w}/dc^2 \leq 0$. Thus,

$$\begin{aligned}
&\frac{d^2}{dc^2}\left[uG(c|u) + \int_0^c U_{t+1}^{c-w} g(w|u)\, dw + (1 - G(c|u)) U_{t+1}^c \right] \\
&\quad \leq g(c|u) \left[\int_0^c \frac{d^2}{dc^2} U_{t+1}^{c-w}\, dw + \frac{d}{dw} U_{t+1}^w \Big|_{w=0} - \frac{d}{dc} U_{t+1}^c \right] \qquad (8) \\
&\quad = g(c|u) \left[\int_0^c \frac{d^2}{dw^2} U_{t+1}^{c-w}\, dw + \frac{d}{dw} U_{t+1}^w \Big|_{w=0} - \frac{d}{dc} U_{t+1}^c \right] = 0.
\end{aligned}$$

Case 2. $u \leq U_{t+1}^c$. Here $u = U_{t+1}^c - U_{t+1}^{c-w_t^c(u)}$. Consequently, the bracket term in (6) becomes

$$U_{t+1}^c - U_{t+1}^{c-w_t^c(u)} G(w_t^c(u)|u) + \int_0^{w_t^c(u)} U_{t+1}^{c-w} g(w|u)\, dw. \qquad (9)$$

Before computing its first and second derivatives, we differentiate the identity $u = U_{t+1}^c - U_{t+1}^{c-w_t^c(u)}$ to obtain an expression for $dw_t^c(u)/dc$ (derivative from the right if $u = U_{t+1}^c$):

$$0 = \frac{d}{dc} U_{t+1}^c - \frac{d}{dw} U_{t+1}^w \Big|_{w=c-w_t^c(u)} \left(1 - \frac{d}{dc} w_t^c(u) \right).$$

Since indeed $dU_{t+1}^w/dw > 0$ in our setup with strictly positive densities, this implies

$$\frac{d}{dc} w_t^c(u) = \frac{\frac{d}{dw} U_{t+1}^w \big|_{w=c-w_t^c(u)} - \frac{d}{dc} U_{t+1}^c}{\frac{d}{dw} U_{t+1}^w \big|_{w=c-w_t^c(u)}}. \qquad (10)$$

By concavity of U^c_{t+1}, its derivative is nonincreasing and hence the identity (10) yields, in particular, $dw^c_t(u)/dc \geq 0$. We now compute the derivatives of (9):

$$\begin{aligned}
&\frac{d}{dc}\bigg[U^c_{t+1} - U^{c-w^c_t(u)}_{t+1} G(w^c_t(u)|u) + \int_0^{w^c_t(u)} U^{c-w}_{t+1} g(w|u)\, dw\bigg] \\
&\qquad = \frac{d}{dc}U^c_{t+1} - \frac{d}{dw}U^w_{t+1}\bigg|_{w=c-w^c_t(u)}\bigg(1-\frac{d}{dc}w^c_t(u)\bigg)G(w^c_t(u)|u) \\
&\qquad\quad - U^{c-w^c_t(u)}_{t+1} g(w^c_t(u)|u)\frac{d}{dc}w^c_t(u) \\
&\qquad\quad + U^{c-w^c_t(u)}_{t+1} g(w^c_t(u)|u)\frac{d}{dc}w^c_t(u) + \int_0^{w^c_t(u)} \frac{d}{dc}U^{c-w}_{t+1} g(w|u)\, dw \\
&\qquad \overset{(10)}{=} \frac{d}{dc}U^c_{t+1} - \frac{d}{dc}U^c_{t+1} G(w^c_t(u)|u) + \int_0^{w^c_t(u)} \frac{d}{dc}U^{c-w}_{t+1} g(w|u)\, dw \\
&\qquad = \frac{d}{dc}U^c_{t+1}\big(1-G(w^c_t(u)|u)\big) + \int_0^{w^c_t(u)} \frac{d}{dc}U^{c-w}_{t+1} g(w|u)\, dw.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{d^2}{dc^2}\bigg[U^c_{t+1} - U^{c-w^c_t(u)}_{t+1} G(w^c_t(u)|u) + \int_0^{w^c_t(u)} U^{c-w}_{t+1} g(w|u)\, dw\bigg] \\
&\quad = \frac{d^2}{dc^2}U^c_{t+1}\big(1-G(w^c_t(u)|u)\big) - \frac{d}{dc}U^c_{t+1} g(w^c_t(u)|u)\frac{d}{dc}w^c_t(u) \\
&\qquad + \frac{d}{dw}U^w_{t+1}\bigg|_{w=c-w^c_t(u)} g(w^c_t(u)|u)\frac{d}{dc}w^c_t(u) + \int_0^{w^c_t(u)} \frac{d^2}{dc^2}U^{c-w}_{t+1} g(w|u)\, dw \\
&\quad \leq g(w^c_t(u)|u)\frac{d}{dc}w^c_t(u)\bigg(\frac{d}{dw}U^w_{t+1}\bigg|_{w=c-w^c_t(u)} - \frac{d}{dc}U^c_{t+1}\bigg) \\
&\qquad + \int_0^{w^c_t(u)} \frac{d^2}{dw^2}U^{c-w}_{t+1} g(w|u)\, dw.
\end{aligned}$$

For the final inequality we used concavity of U^c_{t+1}, as well as $d^2U^{c-w}_{t+1}/dc^2 = d^2U^{c-w}_{t+1}/dw^2$. Noting that (10) implies that $dw^c_t(u)/dc \leq 1$ and once more using concavity of U^c_{t+1}, we may bound the first term from above. Since $g(w|u)$ is nonincreasing in w, we can also bound the second term to obtain

$$\begin{aligned}
&\frac{d^2}{dc^2}\bigg[U^c_{t+1} - U^{c-w^c_t(u)}_{t+1} G(w^c_t(u)|u) + \int_0^{w^c_t(u)} U^{c-w}_{t+1} g(w|u)\, dw\bigg] \\
&\quad \leq g(w^c_t(u)|u)\bigg(\frac{d}{dw}U^w_{t+1}\bigg|_{w=c-w^c_t(u)} - \frac{d}{dc}U^c_{t+1} + \int_0^{w^c_t(u)} \frac{d^2}{dw^2}U^{c-w}_{t+1}\, dw\bigg) = 0.
\end{aligned} \tag{11}$$

Taken together, (8) and (11) establish concavity of the integrand in (6) with respect to c. This implies that U^c_t is concave. Having a second look at the computations just performed reveals that the integrand in (6) has a kink in the second derivative at $u = U^c_{t+1}$.

However, this event has measure zero for any given c, so that we also get that U_t^c is twice continuously differentiable. This completes the induction step. □

Proof of Theorem 3. The main idea of the proof is to translate the problem of setting revenue maximizing prices when w is observable into the problem of maximizing welfare with respect to virtual values (rather than the values themselves), and then to use Lemma 1.

To begin with, note that there is a dual way to describe the policy that maximizes expected welfare under complete information. In the proof of Lemma 1, we characterized it by optimal weight thresholds $w_t^c(u)$. Alternatively, given any requested quantity w (not greater than the remaining c), we may set a valuation per unit threshold $v_t^c(w)$. Requests above this valuation are accepted; those below are not. Such optimal thresholds are characterized by the Bellman-type condition

$$wv_t^c(w) = U_{t+1}^c - U_{t+1}^{c-w}.$$

Thus, one way to write the optimal expected welfare under complete information is

$$\begin{aligned} U_t^c = \int_0^c w \int_{v_t^c(w)}^{\infty} vf(v|w)\,dv\,\bar{f}_w(w)\,dw \\ + \int_0^c \big[\big(1-F(v_t^c(w)|w)\big)U_{t+1}^{c-w} + F(v_t^c(w)|w)U_{t+1}^c\big]\bar{f}_w(w)\,dw. \end{aligned} \tag{12}$$

In contrast, the optimal expected revenue with complete information about w but incomplete information about v satisfies

$$\begin{aligned} R^*(c, T+1-t) = \int_0^c wp_t^c(w)\big(1-F(p_t^c(w)|w)\big)\bar{f}_w(w)\,dw \\ + \int_0^c \big[\big(1-F(p_t^c(w)|w)\big)R^*(c-w, T-t) \\ + F(p_t^c(w)|w)R^*(c, T-t)\big]\bar{f}_w(w)\,dw, \end{aligned} \tag{13}$$

where $p_t^c(w)$ are the per-unit prices from (1). We rephrase this in terms of $\hat{F}$, whose definition requires monotonicity of virtual values. Setting $\hat{v}_t^c(w) := \hat{v}(p_t^c(w), w)$, we have, on the one hand,

$$F(p_t^c(w)|w) = \hat{F}(\hat{v}_t^c(w)|w).$$

On the other hand,

$$\begin{aligned} p_t^c(w)\big(1-F(p_t^c(w)|w)\big) &= \int_{p_t^c(w)}^{\infty} \big[vf(v|w) - (1-F(v|w))\big]\,dv \\ &= \int_{p_t^c(w)}^{\infty} \hat{v}(v,w)\hat{f}(\hat{v}(v,w)|w)\frac{d}{dv}\hat{v}(v,w)\,dv \\ &= \int_{\hat{v}_t^c(w)}^{\infty} \hat{v}\hat{f}(\hat{v}|w)\,d\hat{v}. \end{aligned}$$

Plugging this and the identities for the marginal densities in w into (13) we obtain

$$
\begin{aligned}
&R^*(c, T+1-t)\\
&\quad= \int_0^\infty w \int_{\hat{v}_t^c(w)}^\infty \hat{v} \hat{f}(\hat{v}|w)\, d\hat{v}\, \bar{\hat{f}}_w(w)\, dw\\
&\qquad + \int_0^\infty \big[\big(1-\hat{F}(\hat{v}_t^c(w)|w)\big)R^*(c-w, T-t) + \hat{F}(\hat{v}_t^c(w)|w)R^*(c, T-t)\big]\bar{\hat{f}}_w(w)\, dw.
\end{aligned}
$$

Comparing this with (12), it follows that maximizing expected revenue when w is observable is equivalent to maximizing expected welfare with respect to the distribution of weight and conditional virtual valuation (note the identical zero boundary values at $T+1$). Invoking Lemma 1 applied to $\hat{G}$, we see that $R^*(c, T+1-t)$ is concave with respect to c for all t (note that the fact that the support of virtual valuations contains also negative numbers does not matter for the argument of Lemma 1). □

Proof of Proposition 2. The proposition is an immediate consequence of the characterization (4) of optimal solutions for the w subproblems given $\text{Cap}(w)$, and of a straightforward variational argument ensuring that marginal revenues from marginal increase of $\text{Cap}(w)$ must be constant almost surely in w. □

Proof of Theorem 4. We need to distinguish two cases.

Case 1. Assume that $C > T\int_0^\infty \lambda^{w,*} w\bar{f}_w(w)\, dw$. In this case, $\beta(C, T) = 0$ and $R^d(C, T) = T\int_0^\infty r^w(\lambda^{w,*})w\bar{f}_w(w)\, dw$. We also know that $R^*(C, T) \le R^*(+\infty, T)$, where $R^*(+\infty, T)$ denotes the optimal expected revenue from a stochastic problem without any capacity constraint. But for such a problem, the optimal Markovian policy maximizes at each period the instantaneous expected revenue upon observing w_t, $w_t r^{w_t}(\lambda)$. That is, the optimal policy sets $\lambda_t^{w_t}[+\infty] = \lambda^{w,*}$. Thus,

$$
R^*(C, T) \le R^*(+\infty, T) = T\int_0^\infty w r^w(\lambda^{w,*})\bar{f}_w(w)\, dw = R^d(C, T).
$$

Case 2. Assume now that $C \le T\int_0^\infty \lambda^{w,*} w\bar{f}_w(w)\, dw$. For $\mu \ge 0$, consider the unconstrained maximization problem

$$
\max_{\text{Cap}(\cdot)}\left[\int_0^\infty r^w\left(\frac{\text{Cap}(w)}{Tw\bar{f}_w(w)}\right)Tw\bar{f}_w(w)\, dw + \mu\left(C - \int_0^\infty \text{Cap}(w)\, dw\right)\right].
$$

The Euler–Lagrange equation is $(dr^w/d\lambda)(\text{Cap}(w)/(Tw\bar{f}_w(w))) = \mu$. Hence, if we write $R^d(C, T, \mu)$ for the optimal value of the above problem and if we let $\mu = \beta(C, T)$, where $\beta(C, T)$ is the constant from Proposition 2, then the solution equals the solution of the constrained deterministic problem. In particular, $\int_0^\infty \text{Cap}(w)\, dw = C$ and $R^d(C, T, \beta(C, T)) = R^d(C, T)$.

Recall that for the stochastic problem and for any Markovian policy α, we have

$$
R_\alpha(C, T) = E_\alpha\left[\sum_{t=1}^T w_t p^{w_t}(\lambda_t^{w_t}[c_t]) I_{\{v_t \ge p^{w_t}(\lambda_t^{w_t}[c_t])\}}\right],
$$

and define

$$R_\alpha(C,T,\beta(C,T)) = R_\alpha(C,T) + \beta(C,T)\left(C - E_\alpha\left[\sum_{t=1}^{T} w_t I_{\{v_t \geq p^{w_t}(\lambda_t^{w_t}[c_t])\}}\right]\right).$$

Since for any policy α that is admissible in the original problem, it holds that

$$\sum_{t=1}^{T} w_t I_{\{v_t \geq p^{w_t}(\lambda_t^{w_t}[c_t])\}} \leq C \quad \text{a.s.},$$

we have $R_\alpha(C,T) \leq R_\alpha(C,T,\beta(C,T))$. We show below that, for arbitrary α (which satisfies the capacity constraint or not), it holds that

$$R_\alpha(C,T,\beta(C,T)) \leq R^d(C,T,\beta(C,T)). \tag{14}$$

This yields, for any α that is admissible in the original problem,

$$R_\alpha(C,T) \leq R_\alpha(C,T,\beta(C,T)) \leq R^d(C,T,\beta(C,T)) = R^d(C,T).$$

Taking the supremum over α then concludes the proof for the second case.

It remains to prove (14). The argument uses the filtration $\{\mathcal{F}_t\}_{t=1}^T$ of σ-algebras that contain information prior to time t (in particular the value of c_t) and, in addition, the currently observed w_t:

$$\begin{aligned}
R_\alpha(C,T,\beta(C,T)) &= E_\alpha\left[\sum_{t=1}^{T} w_t\big(p^{w_t}(\lambda_t^{w_t}[c_t]) - \beta(C,T)\big) I_{\{v_t \geq p^{w_t}(\lambda_t^{w_t}[c_t])\}}\right] + \beta(C,T)C \\
&= E_\alpha\left[\sum_{t=1}^{T} E_\alpha\big[w_t\big(p^{w_t}(\lambda_t^{w_t}[c_t]) - \beta(C,T)\big) I_{\{v_t \geq p^{w_t}(\lambda_t^{w_t}[c_t])\}} | \mathcal{F}_t\big]\right] \\
&\quad + \beta(C,T)C \\
&= E_\alpha\left[\sum_{t=1}^{T} w_t\big(p^{w_t}(\lambda_t^{w_t}[c_t]) - \beta(C,T)\big) E_\alpha\big[I_{\{v_t \geq p^{w_t}(\lambda_t^{w_t}[c_t])\}} | \mathcal{F}_t\big]\right] \\
&\quad + \beta(C,T)C \\
&= E_\alpha\left[\sum_{t=1}^{T} w_t\big(r^{w_t}(\lambda_t^{w_t}[c_t]) - \beta(C,T)\lambda_t^{w_t}[c_t]\big)\right] + \beta(C,T)C \\
&\leq E_\alpha\left[\sum_{t=1}^{T} w_t(r^{w_t}(\lambda^{w_t,d}) - \beta(C,T)\lambda^{w_t,d})\right] + \beta(C,T)C \\
&= E_{(w_t)_{t=1}^T}\left[\sum_{t=1}^{T} w_t(r^{w_t}(\lambda^{w_t,d}) - \beta(C,T)\lambda^{w_t,d})\right] + \beta(C,T)C \\
&= T\int_0^\infty (r^w(\lambda^{w,d}) - \beta(C,T)\lambda^{w,d}) w\bar{f}_w(w)\,dw + \beta(C,T)C \\
&= R^d(C,T,\beta(C,T)).
\end{aligned}$$

For the inequality, we have used that $\lambda^{w,d}$ maximizes $r^w(\lambda) - \beta(C,T)\lambda$. □

For the proof of Theorem 5, we first need a lemma.

Lemma 2. *Let $R^{\mathrm{TI}}(C,T)$ be the revenue obtained from the stationary policy α_{TI}. Let $(\widetilde{w}_t, \widetilde{v}_t)_{t=1}^T$ be an independent copy of the process $(w_t, v_t)_{t=1}^T$. Then*

(i)

$$R^{\mathrm{TI}}(C,T) = E_{(w_t)_{t=1}^T}\left[\sum_{t=1}^T r^{w_t}(\lambda^{w_t,d}) w_t \left(1 - P\left[\sum_{s=1}^{t-1} \widetilde{w}_s I_{\{\widetilde{v}_s \ge p^{\widetilde{w}_s,d}\}} > C - w_t\right]\right)\right]$$

(ii)

$$\frac{R^{\mathrm{TI}}(C,T)}{R^d(C,T)} \ge 1 - \frac{\displaystyle\sum_{t=1}^T \int_0^\infty r^w(\lambda^{w,d}) w \left(\min\left(1, \frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w)^2}\right) I^1 + I^2\right) \bar{f}_w(w)\, dw}{\displaystyle T\int_0^\infty r^w(\lambda^{w,d}) w \bar{f}_w(w)\, dw},$$

where $\mu_d := \min(C, T\int_0^\infty \lambda^{w,*} w \bar{f}_w(w)\, dw)/T$, $I^1 = I_{\{w \le (T-t+1)\mu_d\}}$, $I^2 = I_{\{w > (T-t+1)\mu_d\}}$, and $\sigma_d^2 := E[w^2 I_{\{v \ge p^{w,d}\}}] - \mu_d^2 = \int_0^\infty w^2 \lambda^{w,d} \bar{f}_w(w)\, dw - \mu_d^2$.

Proof. (i) Expected revenue $R^{\mathrm{TI}}(C,T)$ may be written as

$$\begin{aligned}
R^{\mathrm{TI}}(C,T) &= E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d} w_t I_{\{v_t \ge p^{w_t,d}\}} I_{\{\sum_{s=1}^{t-1} w_s I_{\{v_s \ge p^{w_s,d}\}} \le C - w_t\}}\right] \\
&= E_{(w_t)_{t=1}^T}\left[\sum_{t=1}^T r^{w_t}(\lambda^{w_t,d}) w_t\right] \\
&\quad - E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d} w_t I_{\{v_t \ge p^{w_t,d}\}} I_{\{\sum_{s=1}^{t-1} w_s I_{\{v_s \ge p^{w_s,d}\}} > C - w_t\}}\right].
\end{aligned}$$

To simplify the second term, we use the fact that v_t and $(w_s, v_s)_{s=1}^{t-1}$ are independent conditional on w_t:

$$\begin{aligned}
&E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d} w_t I_{\{v_t \ge p^{w_t,d}\}} I_{\{\sum_{s=1}^{t-1} w_s I_{\{v_s \ge p^{w_s,d}\}} > C - w_t\}}\right] \\
&\quad = E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T E\big[p^{w_t,d} w_t I_{\{v_t \ge p^{w_t,d}\}} I_{\{\sum_{s=1}^{t-1} w_s I_{\{v_s \ge p^{w_s,d}\}} > C - w_t\}} | w_t\big]\right] \\
&\quad = E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d} w_t E\big[I_{\{v_t \ge p^{w_t,d}\}} | w_t\big] E\big[I_{\{\sum_{s=1}^{t-1} w_s I_{\{v_s \ge p^{w_s,d}\}} > C - w_t\}} | w_t\big]\right] \\
&\quad = E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d} w_t \lambda^{w_t,d} P\left[\sum_{s=1}^{t-1} \widetilde{w}_s I_{\{\widetilde{v}_s \ge p^{\widetilde{w}_s,d}\}} > C - w_t\right]\right]
\end{aligned}$$

$$= E_{(w_t)_{t=1}^T}\left[\sum_{t=1}^T r^{w_t}(\lambda^{w_t,d})w_t P\left[\sum_{s=1}^{t-1}\widetilde{w}_s I_{\{\widetilde{v}_s\geq p^{\widetilde{w}_s,d}\}} > C - w_t\right]\right].$$

This establishes (i).

(ii) Recall that $R^d(C,T) = T\int_0^\infty r^w(\lambda^{w,d})w\bar{f}_w(w)\,dw$. Observe furthermore that $\lambda^{w,d}$ depends on C and T only through the ratio C^{eff}/T, where $C^{\text{eff}} = \min(C, T\int_0^\infty \lambda^{w,*}w\bar{f}_w(w)\,dw)$, via $E[wI_{\{v\geq p^{w,d}\}}] = \int_0^\infty w\lambda^{w,d}\bar{f}_w(w)\,dw = C^{\text{eff}}/T = \mu_d$. Observe first that

$$P\left[\sum_{s=1}^{t-1}\widetilde{w}_s I_{\{\widetilde{v}_s\geq p^{\widetilde{w}_s,d}\}} > C - w_t\right] \leq P\left[\sum_{s=1}^{t-1}\widetilde{w}_s I_{\{\widetilde{v}_s\geq p^{\widetilde{w}_s,d}\}} > T\mu_d - w_t\right]$$

$$= P\left[\sum_{s=1}^{t-1}\widetilde{w}_s I_{\{\widetilde{v}_s\geq p^{\widetilde{w}_s,d}\}} - (t-1)\mu_d > (T-t+1)\mu_d - w_t\right].$$

We trivially bound the last expression by 1 if $(T-t+1)\mu_d - w_t \leq 0$ and otherwise use Chebychev's inequality to deduce

$$P\left[\sum_{s=1}^{t-1}\widetilde{w}_s I_{\{\widetilde{v}_s\geq p^{\widetilde{w}_s,d}\}} - (t-1)\mu_d > (T-t+1)\mu_d - w_t\right]$$

$$\leq P\left[\left(\sum_{s=1}^{t-1}\widetilde{w}_s I_{\{\widetilde{v}_s\geq p^{\widetilde{w}_s,d}\}} - (t-1)\mu_d\right)^2 > ((T-t+1)\mu_d - w_t)^2\right]$$

$$\leq \frac{E\big[\big(\sum_{s=1}^{t-1}\widetilde{w}_s I_{\{\widetilde{v}_s\geq p^{\widetilde{w}_s,d}\}} - (t-1)\mu_d\big)^2\big]}{((T-t+1)\mu_d - w_t)^2} = \frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w_t)^2}.$$

As we are bounding a probability, we can again replace this estimate by the trivial bound 1 whenever it is better, i.e., if w_t is smaller than but close to $(T-t+1)\mu_d$. Thus,

$$E_{(w_t,v_t)_{t=1}^T}\left[\sum_{t=1}^T p^{w_t,d}w_t I_{\{v_t\geq p^{w_t,d}\}} I_{\{\sum_{s=1}^{t-1} w_s I_{\{v_s\geq p^{w_s,d}\}} > C - w_t\}}\right]$$

$$\leq \sum_{t=1}^T\int_0^\infty r^w(\lambda^{w,d})w\left(\min\left(1, \frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w)^2}\right)I_{\{w\leq(T-t+1)\mu_d\}} + I_{\{w>(T-t+1)\mu_d\}}\right)\bar{f}_w(w)\,dw.$$

Finally, dividing by $R^d(C,T)$ yields the desired estimate. □

Proof of Theorem 5. (i) The starting point is the estimate from Lemma 2(ii). Note that $r^w(\lambda^{w,d})w\bar{f}_w(w)$ is an integrable upper bound for

$$r^w(\lambda^{w,d})w\left(\min\left(1, \frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w)^2}\right)I_{\{w\leq(T-t+1)\mu_d\}} + I_{\{w>(T-t+1)\mu_d\}}\right)\bar{f}_w(w).$$

Moreover, for fixed w, for arbitrary $\eta \in (0,1)$, and for $t \leq \eta T$, we have $w < (1-\eta)T\mu_d$ eventually as $T, C \to \infty$, $C/T = \text{const}$. Moreover,

$$\frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w)^2} \leq \frac{\eta T \sigma_d^2}{((1-\eta)T\mu_d - w)^2} \to 0 \quad \text{as } T \to \infty.$$

The Dominated Convergence Theorem implies then that

$$\int_0^\infty r^w(\lambda^{w,d}) w \biggl(\min\biggl(1, \frac{(t-1)\sigma_d^2}{((T-t+1)\mu_d - w)^2}\biggr) I_{\{w \leq (T-t+1)\mu_d\}} + I_{\{w > (T-t+1)\mu_d\}}\biggr) \bar{f}_w(w)\, dw \to 0$$

in the considered limit, for arbitrary $\eta \in (0,1)$ and for $t \leq \eta T$. Consequently, also the term that is subtracted in the estimate Lemma 2(ii) converges to zero.

(ii) A straightforward application of the proof by Gallego and van Ryzin is possible for this last part. For completeness, we spell it out. If w and v are independent, all the $\lambda^{w,d}$ for different w coincide, as do the $\lambda^{w,*}$. Call them λ^d and λ^*, respectively. We have then

$$\begin{aligned} R^{\mathrm{TI}}(C,T) &= p(\lambda^d) E\biggl[\min\biggl(C, \sum_{t=1}^T w_t I_{\{v_t \geq p(\lambda^d)\}}\biggr)\biggr] \\ &= p(\lambda^d) E\biggl[\sum_{t=1}^T w_t I_{\{v_t \geq p(\lambda^d)\}} - \biggl(\sum_{t=1}^T w_t I_{\{v_t \geq p(\lambda^d)\}} - C\biggr)^+\biggr]. \end{aligned}$$

We use now the following estimate for $E[(X-k)^+]$, where X is a random variable with mean m and variance σ^2, and where k is a constant:

$$E[(X-k)^+] \leq \frac{\sqrt{\sigma^2 + (k-m)^2} - (k-m)}{2}.$$

Note that by independence,

$$\begin{aligned} E\biggl[\sum_{t=1}^T w_t I_{\{v_t \geq p(\lambda^d)\}}\biggr] &= E[w] T \lambda^d \\ \mathrm{Var}\biggl[\sum_{t=1}^T w_t I_{\{v_t \geq p(\lambda^d)\}}\biggr] &= T\bigl(E\bigl[\bigl(w I_{\{v \geq p(\lambda^d)\}}\bigr)^2\bigr] - E[w]^2 (\lambda^d)^2\bigr) \\ &= T(E[w^2]\lambda^d - E[w]^2(\lambda^d)^2). \end{aligned}$$

If $\lambda^* T E[w] > C$ and hence if $\lambda^d = C/(TE[w])$, this yields

$$R^{CP}(C,T) \geq R^d(C,T) - p(\lambda^d)\frac{\sqrt{TE[w^2]\lambda^d}}{2} = R^d(C,T)\biggl(1 - \frac{\sqrt{E[w^2]/E[w]}}{2\sqrt{C}}\biggr).$$

If $\lambda^* TE[w] \leq C$ and hence if $\lambda^d = \lambda^*$, then $C \geq E[\sum_{t=1}^T w_t I_{\{v_t \geq p(\lambda^d)\}}]$, so that $E[(\sum_{t=1}^T w_t I_{\{v_t \geq p(\lambda^d)\}} - C)^+] \leq \sqrt{\sigma^2}/2$. Thus,

$$R^{\mathrm{TI}}(C,T) \geq R^d(C,T) - p(\lambda^*)\frac{\sqrt{\lambda^* TE(w^2)}}{2} = R^d(C,T)\left(1 - \frac{\sqrt{E[w^2]/E[w]}}{2\sqrt{\lambda^* E(w) T}}\right).$$

Hence, we can conclude that

$$\frac{R^{\mathrm{TI}}(C,T)}{R^d(C,T)} \geq 1 - \frac{\sqrt{E[w^2]/E[w]}}{2\sqrt{\min(C, T\lambda^* E[w]))}}.$$

□

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