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A wealth-requirement axiomatization of riskiness

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We provide an axiomatic characterization of the measure of riskiness of gambles (risky assets) introduced by Foster and Hart (2009). The axioms are based on the concept of "wealth requirement."

Keywords. Riskiness, gamble, risky asset, reserve, wealth.

JEL classification. D81, G00, G32.

1. Introduction

How are risks evaluated? Here *risk* is meant in the simplest sense: facing certain gains or losses, with given probabilities. The "subjective" approach considers each individual decision-maker separately, and proceeds according to that decision-maker's preference and utility. But can risks be evaluated in an "*objective*" manner—depending only on the risks themselves and not on the specific decision-maker's attitude? While at first sight this may appear to be a tall order, remember that objective measures do exist, as, for example, the return of the gamble (its expectation) and the spread of the gamble (its standard deviation). While the standard deviation is at times used also to measure riskiness, in general it is not a good measure of it. This is so, in particular, since the standard deviation is not monotonic: one may increase the gains and decrease the losses—which clearly lowers the risks—in such a way that the standard deviation actually increases.

New objective measures of riskiness have recently been developed by Aumann and Serrano (2008) and Foster and Hart (2009). While the approach of Aumann and Serrano is axiomatic (it is based mainly on their "duality" axiom), the approach of Foster

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and Hart is constructive, in providing for each gamble the critical wealth level that separates "bad" investments (such as those leading to bankruptcy) from "good" ones (such as those leading to increasing wealth).[1]

In the present paper, we provide an *axiomatic approach to the Foster–Hart measure of riskiness*: we propose four basic axioms for a riskiness measure and show that the minimal function satisfying these axioms is precisely the Foster–Hart measure.

Since "riskiness" does not appear to be a straightforward and obvious concept, one needs to have in mind a certain viewpoint and interpretation. The leading one that we propose is that of *wealth requirement*: the minimal wealth required to engage in a risky activity. Wealth requirements are common, for instance, in high-risk investments (such as hedge funds, in which one should not invest more than a certain proportion of one's wealth), in risky endeavors (such as getting a license for exploration of natural resources—e.g., oil and gas—or for building a large project—e.g., a new transportation system). After all, risks have to do with possible changes in wealth—whether gains or losses—and so it is natural to use the "wealth effects" to measure the riskiness.

The first two axioms that we propose are standard (and are satisfied by the return, the spread, and many other objective measures): the Distribution axiom, which says that only the outcomes and their probabilities matter, and the Scaling axiom, which says, for instance, that doubling the gamble doubles its riskiness (which is measured in the same units as the outcomes). The fact that we are dealing with riskiness and wealth requirements is expressed in the other two axioms: the Monotonicity axiom, which says that decreasing some gains or increasing some losses must increase the wealth requirement, and the Compound Gamble axiom, which says that once the wealth effect is taken into account, the way the gamble is presented does not matter.

Our result is that these axioms characterize the "critical wealth" for a certain class of von Neumann–Morgenstern utility functions, i.e., that wealth level where the decision-maker is indifferent between accepting and rejecting the gamble.[2] Perhaps surprisingly, one of these functions turns out to be *minimal* for all gambles; that is, it bounds from below *all* wealth requirements for *all* gambles (one may thus refer to it as the *critical* critical wealth). This minimal wealth requirement is precisely the Foster–Hart measure of riskiness. In other words, any wealth requirement (that satisfies the axioms) must be at least as conservative as the one given by the Foster–Hart measure.[3]

The paper is organized as follows. In Section 2, we present the formal model and the four axioms. The main result that the Foster–Hart measure is the minimal function satisfying the axioms is stated in Section 3. Our axioms characterize a family of riskiness measures, which turn out to be the critical wealth levels for a certain one-parameter family of utility functions (specifically: CRRA-γ with $\gamma \geq 1$); see Section 4. In Section 5,

[1]An alternative, "ordinal," approach—comparing gambles according to how often they are rejected by risk-averse decision-makers—is provided by Hart (2011): it yields precisely the two orders generated by the Aumann–Serrano and the Foster–Hart measures.

[2]We emphasize that our axioms characterize not only these expected utility functions, but also the resulting "fixed point" where the certainty equivalent of the final wealth (which is the current wealth plus the gamble outcome) equals the current wealth.

[3]For generalizations to sets of gambles and non-expected-utility models, see Michaeli (2012).

we show that dropping the wealth effect from the Compound Gamble axiom yields the Aumann–Serrano index (which may be viewed, in a certain sense, as the *maximal* riskiness function; see Section 7(f)); this is a new axiomatic characterization of the Aumann–Serrano index. An outline of the proofs, together with some additional results, is provided in Section 6. We conclude in Section 7 with discussions and comments on further issues. The proofs and additional material are relegated to the Appendices.

2. The setup

Following Aumann and Serrano (2008) and Foster and Hart (2009), a *gamble*—or "risky asset"—is a real-valued random variable g that represents net changes to the wealth, such that losses are possible and the expected return is positive; i.e.,[4] $\mathbf{P}[g < 0] > 0$ and $\mathbf{E}[g] > 0$. For simplicity, we assume that each gamble g takes only finitely many values, say $x_1, x_2, \ldots, x_m$, with respective probabilities $p_1, p_2, \ldots, p_m$ (where $p_i > 0$ and $\sum_{i=1}^m p_i = 1$); we denote this as $(x_1, p_1; x_2, p_2; \ldots; x_m, p_m)$, or $(x_i, p_i)_{i=1,\ldots,m}$ for short. We denote by $\mathcal{G}$ the collection of all such gambles. For each g in $\mathcal{G}$, let $L(g) := -\min g \equiv -\min_{1\le i\le m} x_i > 0$ be the *maximal loss* of g.

To each gamble g in $\mathcal{G}$, we want to associate a positive number $Q(g)$ that measures its *riskiness*. As stated in the Introduction, an interpretation the reader may want to keep in mind is that $Q(g)$ represents a certain kind of "cushion" or "reserve" needed for g: the *wealth requirement* of g. Let thus[5] $Q : \mathcal{G} \to \mathbb{R}_+$, where the positive number $Q(g) > 0$ is measured in the same units as the outcomes of g.

2.1 *The axioms*

We propose four axioms that a riskiness or wealth requirement should satisfy. In Section 7(b), we provide some further comments on the rationale behind these postulates; let us already say here that a leading concern is that one wants to avoid manipulations that do not affect the gamble but do affect the wealth requirement. From now on, g, h, ... always denote gambles in $\mathcal{G}$.

Distribution. *If g and h have the same distribution, then $Q(g) = Q(h)$.*

That is, only the outcomes and their probabilities matter. We thus no longer distinguish between a gamble and its distribution, and write $g = (x_1, p_1; \ldots; x_m, p_m)$.

Scaling. *$Q(\lambda g) = \lambda Q(g)$ for every $\lambda > 0$.*

That is, $Q(\lambda x_1, p_1; \ldots; \lambda x_m, p_m) = \lambda Q(x_1, p_1; \ldots; x_m, p_m)$. Thus, the wealth requirement does not depend on the unit in which the outcomes are measured: rescaling all outcomes by a factor $\lambda > 0$ rescales it by the same λ.

[4]We write **P** for probability and **E** for expectation. Since, as we will see shortly, only the distribution of the random variables matter, there is no need to specify the underlying probability spaces. Alternatively, take a (large enough) probability space over which all the gambles are defined.

[5]We write $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = (0, \infty)$ for the set of reals and the set of positive reals, respectively.

MONOTONICITY. *If*[6] $g \geq h$ *and* $g \neq h$*, then* $Q(g) < Q(h)$*.*

Thus, if the outcomes of h are less than or equal to the outcomes of g (i.e., the gains in h can only be less than those in g and the losses can only be greater)—with some inequalities being strict—then the wealth requirement for h must be strictly greater than the wealth requirement for g. In terms of distributions, one can write this as $Q(x_1 + \delta, p_1; x_2, p_2; \ldots; x_m, p_m) < Q(x_1, p_1; x_2, p_2; \ldots; x_m, p_m)$ for any $\delta > 0$ (iterating this condition yields, together with the Distribution axiom, the same condition for any $g \gneqq h$). Clearly, Monotonicity together with Distribution imply that the function Q is monotonically decreasing with respect to first-order stochastic domination (see Proposition 14 in Appendix A.3.1).

While the axioms up to now are standard, the final one—the Compound Gamble axiom—is the main one that embodies the idea of "wealth requirement." We illustrate it with a simple example.

Let g be a gamble with two possible outcomes, \$200 and −\$100 (i.e., a gain of \$200 or a loss of \$100), with probabilities p_1 and $p_2 = 1 - p_1$, respectively, and assume that its wealth requirement is, say, $Q(g) = \$500$. Consider two other gambles, h_1 and h_2, that are independent of g, and let f be the compound gamble consisting of g followed by h_1 when g has resulted in a gain of \$200, and g followed by h_2 when g has resulted in a loss of \$100 (see Figure 1, top). Assume that the wealth requirement of h_1 is $Q(h_1) = \$700$, which equals the original wealth requirement of g of $\$500 = Q(g)$ plus the gain of g of \$200 realized before h_1 is taken; assume also that the wealth requirement of h_2 is $Q(h_2) = \$400$, which equals the wealth requirement of g minus the \$100 loss realized before h_2 is taken. Thus in each case, the "new" wealth requirement—which equals the "old" wealth requirement $Q(g)$ of g plus the outcome x of g—is precisely the correct wealth requirement for the continuation, h_1 or h_2. What the Compound Gamble axiom says is that the original wealth requirement of g of \$500 is appropriate also for the compound gamble f. This may be viewed as a kind of *sure-thing principle*: if no matter what the outcome of g will be—i.e., ex post—the resulting wealth requirement will be just right to continue, then the wealth requirement is the right one also overall—i.e., ex ante. In other words, one does not need to know what the outcome of g will be, since one will always have the correct wealth requirement to continue (with either h_1 or h_2, respectively), no matter what the outcome of g will be.

The Compound Gamble axiom is slightly more general: it allows for no continuation gamble in some instances. For example, consider the compound gamble f' where the same h_1 is taken after the \$200 gain as in f, but there is no further gamble after the \$100 loss (see Figure 1, bottom). Since, again, whenever a gamble is taken there is always the right wealth requirement, it follows that the wealth requirement for f' is also the same[7] $\$500 = Q(g)$.

To state the axiom formally, let $\mathbf{1}_A$ denote the indicator of an event A (i.e., $\mathbf{1}_A$ is a random variable that equals 1 if A occurs and 0 otherwise) and let $[g = x]$ denote the

[6]This presumes that g and h are defined on the same probability space Ω. The condition is thus $g(\omega) \geq h(\omega)$ for every $\omega \in \Omega$, with strict inequality for some $\omega \in \Omega$.

[7]In the extreme case where there are no continuation gambles at all, $f = g$ and so, of course, $Q(f) = Q(g)$.

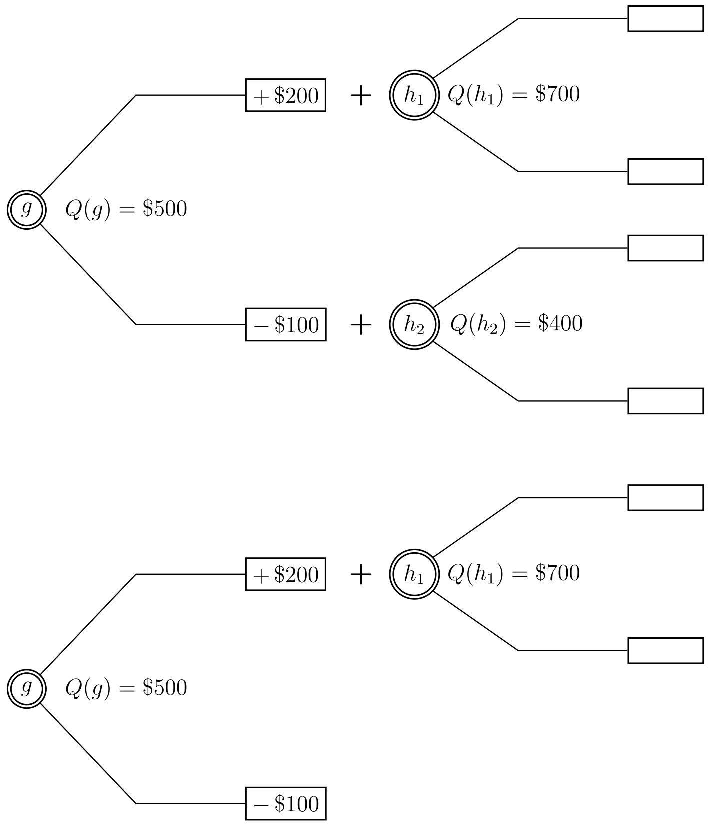

FIGURE 1. The compound gambles f (top) and f' (bottom).

event $\{\omega : g(\omega) = x\}$ that the value of g is x. The compound gambles above can thus be written as $f = g + \mathbf{1}_{[g=\$200]}h_1 + \mathbf{1}_{[g=-\$100]}h_2$ and $f' = g + \mathbf{1}_{[g=\$200]}h_1$. Write $g|_A$ for the restriction of g on A; thus $g|_A \equiv x$ means that $g(\omega) = x$ for all $\omega \in A$, or $A \subset [g = x]$. Finally, h is independent of A if the random variables h and $\mathbf{1}_A$ are independent; that is,[8] $\mathbf{P}[h = y|A] = \mathbf{P}[h = y]$ for every value y of h. Since one can apply the axiom repeatedly, we state it in its simplest form.

COMPOUND GAMBLE. *Let $f = g + \mathbf{1}_A h$ be a compound gamble, where g, $h \in \mathcal{G}$ and A is an event such that g is constant on A, i.e., $g|_A \equiv x$ for some x, and h is independent of A. If $Q(h) = Q(g) + x$, then*[9] $Q(f) = Q(g)$.

For example, if $g = (x_1, p_1; x_2, p_2; \ldots; x_m, p_m)$, $h = (y_1, q_1; y_2, q_2; \ldots; y_k, q_k)$, and $A = [g = x_1]$ is the event that g takes the value x_1, then $f = (x_1 + y_1, p_1q_1; x_1 + y_2, p_1q_2; \ldots; x_1 + y_k, p_1q_k; x_2, p_2; \ldots; x_m, p_m)$. In the example of Figure 1, bottom, the Compound Gamble axiom is applied once; in Figure 1, top, it is applied twice.

[8]The event A is always assumed to have positive probability, i.e., $\mathbf{P}[A] > 0$, and so the conditional probabilities are well defined.

[9]To see that f is in $\mathcal{G}$: first, $\mathbf{E}[f] = \mathbf{E}[g] + \mathbf{P}[A]\mathbf{E}[h] > 0$; and second, a negative value of f is $-L(g)$ when $x \neq -L(g)$ and $-L(g) - L(h)$ when $x = -L(g)$.

3. The main result

Let $\mathbf{R}$ be the *measure of riskiness*, introduced by Foster and Hart (2009): for every gamble $g \in \mathcal{G}$, the riskiness $\mathbf{R}(g)$ of g is given by[10]

$$\mathbf{E}\left[\log\left(1 + \frac{1}{\mathbf{R}(g)} g\right)\right] = 0. \quad (1)$$

Our main result is the following theorem.

Theorem 1 (Main Theorem). *The* minimal *function that satisfies the four axioms the Distribution, Scaling, Monotonicity, and Compound Gamble axioms is the measure of riskiness* $\mathbf{R}$. *More precisely: (i) the function* $\mathbf{R}$ *satisfies these four axioms; and (ii) if a function* Q *satisfies these four axioms, then either* $Q(g) = \mathbf{R}(g)$ *for all* $g \in \mathcal{G}$, *or* $Q(g) > \mathbf{R}(g)$ *for all* $g \in \mathcal{G}$.

The Main Theorem says that the *minimal* wealth requirement is precisely our riskiness measure $\mathbf{R}$. As we see in the next section, there are other possible wealth requirement functions Q that satisfy the axioms; however, each one yields *strictly higher* wealth requirement levels for *all* gambles.

We may now combine this result with that of Foster and Hart (2009). A function Q that associates a positive number with each gamble may be used to decide at which wealth levels to accept or reject gambles. For instance, consider a decision-maker who accepts gambles whenever his wealth is no less than the wealth requirement given by Q; i.e., he accepts a gamble g at wealth w if and only if $w \geq Q(g)$; call him a *Q-decision-maker*. As in Foster and Hart (2009), a decision-maker is *guaranteed no bankruptcy* if, for any sequence of gambles, his acceptance and rejection decisions make his wealth never go to zero (with probability 1). The result of Theorem 1 in Foster and Hart (2009) together with Theorem 1 above immediately yield a corollary.

Corollary 2. *If Q satisfies the four axioms, then a Q-decision-maker is guaranteed no bankruptcy.*

4. The other riskiness measures

What happens if we drop the *minimality* requirement? It turns out that our four axioms characterize a one-parameter family of functions (the minimal one being the Foster–Hart measure $\mathbf{R}$). Rewriting equation (1), which determines $\mathbf{R}(g)$, as

$$\mathbf{E}\big[\log(\mathbf{R}(g) + g)\big] = \log(\mathbf{R}(g))$$

shows that $\mathbf{R}(g)$ is that wealth level where a decision-maker with utility function $\log(x)$ is indifferent between accepting and rejecting g (at all higher wealths, he accepts g; at all lower wealths, he rejects g). Call this the *critical wealth* of the log utility for g.

[10]More precisely, the equation $\mathbf{E}[\log(1 + (1/r)g)] = 0$ (for $r > L(g)$, since $\log x$ is only defined for $x > 0$) has a unique solution $r \equiv \mathbf{R}(g)$; see Foster and Hart (2009, Lemma 9). When $g = (x_1, p_1; \dots; x_m, p_m)$, (1) becomes $\sum_{i=1}^{m} p_i \log(1 + x_i/\mathbf{R}(g)) = 0$. We emphasize that while (1) (as well as (3) and (4) below) may appear at first sight to be a sort of expected utility representation of $\mathbf{R}(g)$, this is *not* the case; (1) is an *implicit* equation defining it (we thank Moti Michaeli for this observation).

Now $\log(x)$ is precisely the utility function $u(x)$ with *Constant Relative Risk Aversion* (CRRA) equal to 1; i.e., its Arrow–Pratt coefficient of Relative Risk Aversion, $\mathrm{RRA}_u(x) := -xu''(x)/u'(x)$, satisfies $\mathrm{RRA}_u(x) = 1$ for all $x > 0$; see Arrow (1965, 1971) and Pratt (1964). More generally, the CRRA-γ utility function, which satisfies $\mathrm{RRA}_u(x) = \gamma$ for all $x > 0$, is[11] $u_\gamma(x) = (1-\gamma)x^{1-\gamma}$ for $\gamma \neq 1$ and $u_1(x) = \log(x)$ for $\gamma = 1$. The *critical wealth* of CRRA-γ for a gamble g, which we denote[12] $R_\gamma(g)$, is that wealth level where CRRA-γ is indifferent between accepting and rejecting g; it is thus given by the equation

$$\mathbf{E}\big[u_\gamma(R_\gamma(g)+g)\big] = u_\gamma(R_\gamma(g)). \tag{2}$$

As we see below (Lemmata 5 and 8), for each $\gamma \geq 1$ and gamble g in $\mathcal{G}$, the positive number $R_\gamma(g)$ is well defined by (2) and, moreover, $R_\gamma(g)$ is strictly increasing in γ (this is due to CRRA-γ becoming more risk-averse—i.e., more "conservative"—as γ increases, and so the corresponding critical wealth is getting higher). We call the function R_γ the *γ-riskiness measure*: for $\gamma = 1$, it is equivalently given by (1)—and so $R_1 \equiv \mathbf{R}$, the Foster–Hart measure—and for $\gamma > 1$, it is given by

$$\mathbf{E}\left[\left(1+\frac{1}{R_\gamma(g)}g\right)^{1-\gamma}\right] = 1. \tag{3}$$

Our result is that the four axioms precisely characterize the family R_γ, the critical wealth levels of CRRA-γ, for $\gamma \geq 1$.

Theorem 3. *A function Q satisfies the four axioms of the Main Theorem if and only if there exists $\gamma \geq 1$ such that $Q(g) = R_\gamma(g)$ for all $g \in \mathcal{G}$.*

Thus, our axioms imply that each wealth requirement function is determined by a von Neumann–Morgenstern utility that has a constant coefficient of relative risk aversion of at least 1; the wealth requirement is then the minimal wealth level where the gamble is accepted by this utility.

To illustrate the connection between (2) and the Compound Gamble axiom, recall the example of Figure 1 in Section 2.1. Using (2) for g and for h (in the first and third equalities below, respectively), we get (put $u \equiv u_\gamma$)

$$\begin{aligned} u(500) &= \mathbf{E}[u(500+g)] \\ &= p_1 u(700) + (1-p_1)u(400) \\ &= p_1 \mathbf{E}[u(700+h_1)] + (1-p_1)u(400) \\ &= \mathbf{E}[u(500+f')], \end{aligned}$$

i.e., (2) for the compound gamble f'.

[11] Up to positive linear transformations, which do not affect what we do (see (2)); in the proofs in Appendix A, it will be convenient to use a different version of u_γ (see (6)).

[12] In Aumann and Serrano (2008, Section IV.C), $R_\gamma(g)$ is denoted $w_\gamma(g)$.

5. Wealth independence and the Aumann–Serrano index

To understand the role of "wealth," note that it appears only in the Compound Gamble axiom. There the change in wealth due to the outcome x of the first gamble g affects the Q-requirement for the second gamble h, i.e., $Q(h) = Q(g) + x$; if, for instance, g resulted in a gain (i.e., $x > 0$), then one can afford an h with a higher Q-requirement. What happens if we make the measure Q "wealth independent" instead? That is, replace $Q(h) = Q(g) + x$ with $Q(h) = Q(g)$, which yields the following axiom.

Wealth-Independent Compound Gamble. *Let $f = g + \mathbf{1}_A h$ be a compound gamble, where $g, h \in \mathcal{G}$ and A is an event such that g is constant on A, i.e., $g|_A \equiv x$ for some x, and h is independent of A. If $Q(h) = Q(g)$, then $Q(f) = Q(g)$.*

Using this axiom instead of the Compound Gamble axiom characterizes the Aumann–Serrano index[13] R^{AS}:

Theorem 4. *A function Q satisfies the Distribution, Monotonicity, Scaling, Wealth-Independent Compound Gamble axioms if and only if Q is proportional to the Aumann–Serrano index of riskiness R^{AS} (i.e., there exists a constant $c > 0$ such that $Q(g) = cR^{\text{AS}}(g)$ for all $g \in \mathcal{G}$).*

The *index of riskiness* developed by Aumann and Serrano (2008)[14] is uniquely determined by the equation[15]

$$\mathbf{E}\left[\exp\left(-\frac{1}{R^{\text{AS}}(g)}g\right)\right] = 1. \tag{4}$$

How can one interpret the Wealth-Independent Compound Gamble axiom? One can show that it implies $Q(\sum_{t=1}^{T} g_t) = Q(g_1)$, where the g_t are independent and identically distributed gambles (see Proposition 22 in Appendix B for a general statement). This suggests[16] that such a Q may represent a certain "acceptable level of riskiness" that remains fixed over time, regardless of the gambles already taken, their realizations, and the resulting changes in wealth (see also Aumann and Serrano 2008, Section V.H). For instance, this level of riskiness could correspond to acceptance of g at *all* wealth levels (cf. Hart 2011)—again, a "wealth independence" requirement.

Returning to the class of γ-riskiness functions R_γ (for $\gamma \geq 1$) that are given by Theorem 3, the minimal one is $R_1 \equiv \mathbf{R}$. What is the *maximal* one—the *most conservative* riskiness measure?

For every gamble g, the sequence $R_\gamma(g)$ increases with γ (since a higher γ corresponds to higher risk aversion and, thus, to a higher critical wealth level); moreover, it

[13]Up to rescaling by a constant $c > 0$ (minimality does not help here).

[14]This index was used in the technical report of Palacios-Huerta, Serrano, and Volij (2004); see the footnote on page 810 of Aumann and Serrano (2008). It is the inverse of the "adjustment coefficient" of the insurance risk literature; see Meilijson (2009).

[15]We write $\exp(x)$ for e^x.

[16]Proposed by Robert Aumann (personal communication).

can be shown that $R_\gamma(g) \to \infty$ as $\gamma \to \infty$ and so the maximal riskiness is infinite. Nevertheless, the functions R_γ have a well-defined limit behavior as $\gamma \to \infty$, which turns out to be related to the Aumann–Serrano index. Specifically, for every $g \in \mathcal{G}$, we have

$$\lim_{\gamma \to \infty} \frac{1}{\gamma} R_\gamma(g) = R^{\mathrm{AS}}(g) \tag{5}$$

(this can be easily shown by comparing (3) with (4); see also Aumann and Serrano 2008, Theorem C).

Equivalently,

$$\lim_{\gamma \to \infty} \frac{R_\gamma(g)}{R_\gamma(h)} = \frac{R^{\mathrm{AS}}(g)}{R^{\mathrm{AS}}(h)} = \frac{Q^{\mathrm{AS}}(g)}{Q^{\mathrm{AS}}(h)}$$

for any $Q^{\mathrm{AS}} = cR^{\mathrm{AS}}$ with $c > 0$. Thus any Q^{AS} that satisfies the axioms of Theorem 4 yields *riskiness comparisons* that are the same as those of the most conservative Q that satisfies the axioms of the Main Theorem; i.e.,

$$R^{\mathrm{AS}}(g) > R^{\mathrm{AS}}(h) \quad \text{if and only if} \quad R_\gamma(g) > R_\gamma(h) \quad \text{for all } \gamma \textit{ large} \text{ enough.}$$

The Aumann–Serrano index may thus be viewed as the *maximal* riskiness in the family of measures R_γ of Theorem 3, whereas the Foster–Hart measure is the *minimal* one (Theorem 1). See also Section 7(f).

6. An outline of the proofs

This section is devoted to an outline of our proofs: we state a number of intermediate results that may be of interest on their own, and provide an informal tour of the proofs. The formal proofs are relegated to the Appendices.

6.1 *Proof outline for Theorems 1 and 3*

We start with Theorem 3, which says that Q satisfies the four axioms of Section 2.1 if and only if $Q = R_\gamma$ for some $\gamma \geq 1$. This is the combination of the following three results:

Lemma 5. *For each $\gamma \geq 1$, the function R_γ is well defined by (2).*

Proposition 6. *For each $\gamma \geq 1$, the function R_γ satisfies the four axioms.*

Proposition 7. *If Q satisfies the four axioms, then there exists $\gamma \geq 1$ such that $Q = R_\gamma$.*

While the proofs of Lemma 5 and Proposition 6 are essentially straightforward, the proof of Proposition 7 is not. We describe the latter in Section 6.2.

The next lemma (whose proof is also easy) shows that the R_γ functions are "well ordered."

Lemma 8. *$R_\gamma(g)$ strictly increases with γ, i.e., $R_\beta(g) < R_\gamma(g)$ for every $g \in \mathcal{G}$ and $1 \le \beta < \gamma$.*

So $R_1 \equiv \mathbf{R}$ is the minimal R_γ for $\gamma \ge 1$; hence, by Theorem 3, it is the minimal Q that satisfies the four axioms—which proves our Main Theorem.

6.2 *Proof outline for Proposition 7*

The main difficulty lies in the proof of Proposition 7, which we outline now. We first provide a number of consequences of our axioms.

The first one says that Q is always greater than the maximal loss.

Proposition 9. *If Q satisfies the four axioms, then $Q(g) > L(g)$ for every g.*

The proof here is somewhat tricky.[17]

Second, take a sequence of independent gambles $g_1, g_2, \ldots, g_t, \ldots$ with $Q(g_t) = 1$ for all t. Let $x_1, \ldots, x_m$ be the distinct values of g_1 and let f_2 be the compound gamble consisting of g_1 followed, after each outcome x_j, by the $(1 + x_j)$-multiple of g_2. Thus,

$$\begin{aligned} f_2 &= g_1 + \sum_{j=1}^{m} \mathbf{1}_{[g_1 = x_j]}(1 + x_j) g_2 = g_1 + (1 + g_1) g_2 \\ &= (1 + g_1)(1 + g_2) - 1. \end{aligned}$$

Since $Q((1 + x_j)g_2) = (1 + x_j)Q(g_2) = 1 + x_j$ by the Scaling axiom, applying the Compound Gamble axiom m times yields $Q(f_2) = 1$. In the same way, from f_2 and g_3, we get

$$f_3 = (1 + f_2)(1 + g_3) - 1 = (1 + g_1)(1 + g_2)(1 + g_3) - 1$$

with $Q(f_3) = 1$ and so on, leading to the following condition.

- If $1 + f_N = \prod_{t=1}^{N}(1 + g_t)$, where $(g_t)_{t=1,2,\ldots}$ is a sequence of independent gambles with $Q(g_t) = 1$ for all t and N is a positive integer, then $Q(f_N) = 1$.

Since the Compound Gamble axiom allows for no gamble after some outcomes, we can get a more general property. To state it formally, it is convenient to use the notion of a *stopping time* T, which is a random variable whose values are positive integers that specify how many gambles we take. The requirement is that T be *adapted to the sequence* $(g_t)_t$, which means that for each integer n, the event $[T = n]$ that we stop immediately after g_n is determined by the outcomes of $g_1, \ldots, g_n$ only (i.e., by the past and not the future). We get the following condition.

Multiplicative Compounding. *If $1 + f_T = \prod_{t=1}^{T}(1 + g_t)$, where $(g_t)_{t=1,2,\ldots}$ is a sequence of independent gambles with $Q(g_t) = 1$ for all t and T is a bounded stopping time adapted to the sequence $(g_t)_t$, then $Q(f_T) = 1$.*

[17] In fact, it was a stumbling block for a long time, and, in previous versions of the paper, the statement of Proposition 9 appeared as an additional axiom.

Proposition 10. *If Q satisfies the four axioms, then Q satisfies the Multiplicative Compounding condition.*

Third, recall that $R_\gamma(g)$ is that wealth level where CRRA-γ is indifferent between accepting and rejecting g; see (2). When $\gamma \geq 1$, this critical wealth is well defined (see Lemma 5), since each gamble is rejected at low enough wealth and accepted at high enough wealth. This is, however, no longer true when $0 < \gamma < 1$: for each such γ, there are gambles in $\mathcal{G}$ that CRRA-γ *accepts at all wealth levels* (and so in this case, (2) has no solution). For example, the gamble $g = (4, 1/2; -1, 1/2)$ is always accepted by the CRRA-1/2 utility $u_{1/2}(x) = \sqrt{x}$, since $(1/2)\sqrt{w+4} + (1/2)\sqrt{w-1} > \sqrt{w}$ for all $w \geq 1$. However, if a critical wealth does exist, it is necessarily unique, and so we extend the definition of $R_\gamma(g)$ to all $\gamma > 0$: it is given by (2) whenever it has a solution (and thus for $0 < \gamma < 1$, it is defined only for some[18] g).

From (2), it can be shown that R_γ satisfies the Multiplicative Compounding condition also for $0 < \gamma < 1$ (for $\gamma \geq 1$, recall Propositions 6 and 10). Moreover, as γ increases, $R_\gamma(g)$ increases continuously from $L(g)$ to ∞ (this generalizes Lemma 8) and we have the following lemma.

Lemma 11. *For every $g \in G$ with $L(g) < 1$, there exists a unique $\gamma \equiv \gamma_g > 0$ such that $R_\gamma(g) = 1$.*

(This is stated as Lemma 16 in Appendix A.3.4.) We emphasize that this γ need not satisfy $\gamma \geq 1$.

We are now ready to present the basic argument that proves Proposition 7. Fix a function Q that satisfies the four axioms. Let g be a gamble with $Q(g) = 1$, and apply Lemma 11 (recall Proposition 9) to get $\gamma > 0$ such that $R_\gamma(g) = 1$. Take a sequence $(g_t)_t$ of independent and identically distributed (i.i.d.) gambles, all with the same distribution as g. For every bounded stopping time T, the gamble $f \equiv f_T = \prod_{t=1}^{T}(1 + g_t) - 1$ satisfies $Q(f) = 1$ and also $R_\gamma(f) = 1$ (see Proposition 10 and the paragraph just before Lemma 11 above). Now take z_1 very large, and $z_2 > -1$ and very close to -1; then there is a stopping time T for which the resulting f has, with high probability, values that are close to either z_1 or z_2 (the sequence f_n is a multiplicative random walk, and we put two absorbing barriers at z_1 and z_2). For simplicity, ignore the various technical approximation issues,[19] and assume that T is bounded and that f takes only these two values z_1 and z_2.

Next, take another g' gamble with $Q(g') = 1$, and apply Lemma 11 to get γ' such that $R_{\gamma'}(g) = 1$. Proceed as in the previous paragraph: starting with an i.i.d. sequence $(g'_t)_t$ with g'_t having the same distribution as g', we get a stopping time T' and a resulting $f' \equiv f'_{T'}$ that takes only the two values z_1 and z_2, the same ones as f (we again ignore the approximation issues) for which we have $Q(f') = 1$ and $R_{\gamma'}(f') = 1$. Since f and f' take

[18] All statements below regarding R_γ for $\gamma < 1$ should thus be understood to hold whenever R_γ is defined.

[19] The Continuity requirement of Q could be of help here; however, we did not require it as an axiom (and it does not hold in full generality; cf. Foster and Hart 2009, Section V.B). We use the Monotonicity axiom instead, by bounding from above and from below as needed.

the same two values, and $Q(f) = 1 = Q(f')$, it follows that f and f' must have the same distribution (otherwise one would dominate the other[20] and then, by the Distribution and Monotonicity axioms, we would have either $Q(f) < Q(f')$ or $Q(f') < Q(f)$), and so $R_{\gamma'}(f) = R_{\gamma'}(f') = 1$. But we also have $R_\gamma(f) = 1$, which implies, by Lemma 11, that $\gamma' = \gamma$.

What we have thus shown is that there is a unique γ^* such that $Q(g) = 1$ implies that $R_{\gamma^*}(g) = 1$. From this, it follows (by the Scaling axiom applied to both functions) that $Q \equiv R_{\gamma^*}$. But Q must be defined for all gambles g, whereas, as we have seen, this is not the case with R_γ when $0 < \gamma < 1$; therefore $\gamma^* \geq 1$, completing the proof of Proposition 7.

6.3 *Proof outline for Theorem 4*

The proof of Theorem 4 is similar to the above proof (and in some ways simpler). One change is that when we iterate the Wealth-Independent Compound Gamble axiom, we get the following condition.

Additive Compounding. *If $f_T = \sum_{t=1}^{T} g_t$, where $(g_t)_{t=1,2,\ldots}$ is a sequence of independent gambles with $Q(g_t) = 1$ for all t and T is a bounded stopping time adapted to the sequence $(g_t)_t$, then $Q(f_T) = 1$.*

Proposition 12. *If Q satisfies the Distribution and Wealth-Independent Compound Gamble axioms, then Q satisfies the Additive Compounding condition.*

The proof of Proposition 12 is parallel to the proof of Proposition 10 (see Section 6.2 above). Using the Additive Compounding condition together with the appropriate stopping times then shows that if $Q(g) = 1 = cR^{\mathrm{AS}}(g)$ and $Q(g') = 1 = c'R^{\mathrm{AS}}(g')$, then necessarily $c = c'$ (this is the counterpart of showing that $\gamma = \gamma'$ at the end of Section 6.2), which proves Theorem 4.

7. Discussion

In this section we provide a number of additional comments.

(a) Nonmanipulability interpretations of the axioms. The axioms may be viewed as certain conditions that make the wealth requirement immune to "manipulations."

Take the Scaling axiom. If, for instance, there is a gamble g such that the wealth requirement $Q(g/2)$ for $g/2$ is less than $1/2$ the requirement for g, i.e., $Q(g/2) < Q(g)/2$, then one can lower the requirement for g by splitting it into two parts,[21] $g_1 = g_2 = g/2$, and then $Q(g_1) + Q(g_2) < Q(g_1 + g_2) = Q(g)$. If, instead, $Q(g/2) > Q(g)/2$, then two institutions each holding $g/2$ can lower their wealth requirement from $Q(g/2)$ to $Q(g)/2$ by presenting a "merged" exposure to g and splitting the wealth requirement. More generally, dividing g into λg and $(1-\lambda)g$ should not affect the wealth requirement, so we

[20] In the sense of first-order stochastic dominance.

[21] Thus, g_1 and g_2 are identical random variables: in each state of the world, both of them lose or gain exactly the same amount; the risks of g_1 and g_2 are thus fully correlated.

must have $Q(g) = Q(\lambda g) + Q((1-\lambda)g)$ for every $0 < \lambda < 1$, which yields the Scaling axiom. Since λg and $(1-\lambda)g$ are fully correlated risks, Scaling essentially says that the wealth requirement is additive over such fully correlated risks.

Next, the Compound Gamble axiom may be viewed as saying that the wealth requirement should not be affected by the way the gamble is presented, whether as a two-step gamble (g followed by h) or a one-step gamble[22] (f).

Finally, consider the Monotonicity axiom. If it is not satisfied, then one can keep the wealth requirement the same, or even lower it, while making the gamble worse by decreasing gains and/or increasing losses. Most existing riskiness measures—such as the standard deviation or the Value-at-Risk (VaR)—suffer from this very significant drawback, which allows easy manipulations of the risks involved without affecting the required wealth or the needed reserves.[23] We emphasize that these manipulations are possible even when there is Weak Monotonicity (i.e., $Q(g) \leq Q(h)$ instead of $Q(g) < Q(h)$; see Appendix A.3.2 and Appendix C). Note that VaR does satisfy Weak Monotonicity.

(b) The axioms are indispensable. In Appendix C, we show that the axioms are indispensable for our result: dropping any one of them while keeping the others allows for additional functions Q. In particular, Proposition 23 in Appendix C shows what happens when one replaces Monotonicity with Weak Monotonicity. Moreover, in Appendix D, we prove that if we replace the Scaling axiom by a certain the Continuity requirement, then the axioms characterize the critical wealth level for a class of utility functions that is larger than CRRA.

(c) Relative returns. In view of the Scaling axiom, one may restate everything in terms of *relative* rather than *absolute* returns. Indeed, the net absolute returns described by a gamble g at wealth level w yield net relative returns of g/w (and gross relative returns of $1 + g/w$). Then, instead of dealing with the "right" wealth for the gamble g, one deals with the "right" *proportion* of g per unit of wealth. The two approaches are thus equivalent.[24,25]

(d) Maximal growth rate (the "Kelly criterion"). While our axioms lead to the log utility, we emphasize that a decision-maker with log utility does *not* behave according to **R**. A log-utility maximizer takes the gamble g when his wealth equals $K \equiv K(g)$, which maximizes $\mathbf{E}[\log(1 + g/K)]$ (and not when his wealth is $\mathbf{R}(g)$, where he is indifferent between accepting and rejecting g). Following Kelly (1956) and the subsequent extensive literature (see Foster and Hart 2009, Section IV.E, point 5), the rule induced by this function $K(g)$ (called the "Kelly function") *maximizes the growth rate* of one's wealth.

Now the first-order condition that determines $K \equiv K(g)$ is $\mathbf{E}[(1+g/K)^{-1}(-g/K^2)] = 0$, which is equivalent to $\mathbf{E}[(1+g/K)^{-1}] = 1$ (multiply by K and subtract from 1). Comparing this equation with (3) shows that it is precisely the equation for R_2 (i.e., when

[22] At first sight, one may ask why do we not take $f = g + h$? The reason is that after g has been realized, the wealth has changed and so different gambles can now be afforded (cf. the discussion in Section 5 above).

[23] For instance, any losses that are greater than VaR can be arbitrarily increased, which yields a significantly riskier gamble, but does not affect VaR.

[24] In fact, Multiplicative Compounding can be interpreted as compounding the *relative* returns.

[25] However, they are not equivalent in the Aumann–Serrano framework; see Schreiber (2012).

$\gamma = 2$) and so $K(g) = R_2(g)$ for all g; i.e., K is the *critical* wealth for the CRRA-2 utility $u_2(x) = -1/x$. The Kelly function is thus one of our riskiness functions—albeit not the minimal one $\mathbf{R}$; in particular, it satisfies the four axioms and all their consequences (see (e) below).[26]

(e) Additional properties. It is straightforward to check that all the properties of $\mathbf{R}$ in Section V of Foster and Hart (2009)—such as subadditivity, convexity, dilution, independent gambles, and continuity—are also satisfied by all the R_γ with $\gamma \geq 1$; indeed, all the proofs there work mutatis mutandis with u_γ instead of $u_1 \equiv \log$. Theorem 3, therefore, implies that all these properties follow from our four axioms Distribution, Monotonicity, Scaling, and Compound Gamble.

(f) The Aumann–Serrano index as the maximal riskiness. As we saw in Section 5, the Aumann–Serrano index may be viewed as the "asymptotic maximal riskiness." In terms of axioms, consider the functions $\tilde{R}_\gamma := R_\gamma/\gamma$; these rescaled versions of the R_γ functions satisfy Distribution, Scaling, and Monotonicity, and the following appropriately rescaled version of Compound Gamble:

δ-COMPOUND GAMBLE. *Let $f = g + \mathbf{1}_A h$ be a compound gamble, where $g, h \in \mathcal{G}$ and A is an event such that g is constant on A, i.e., $g|_A \equiv x$ for some x, and h is independent of A. If $Q(h) = Q(g) + \delta x$, then $Q(f) = Q(g)$.*

(The change is in the condition $Q(h) = Q(g) + \delta x$.) Clearly, $\tilde{R}_\gamma$ satisfies this axiom for $\delta = 1/\gamma$.

As δ decreases to 0, so does the wealth effect (i.e., the outcome x of g multiplied by δ), leading in the limit to the Wealth-Independent Compound Gamble axiom (which is nothing but the 0-Compound Gamble axiom). This is the axiomatic counterpart of (5): $\tilde{R}_\gamma = R_\gamma/\gamma$ converges to the Aumann–Serrano index R^{AS}, which is indeed independent of wealth.

(g) Bounded gambles. As in Aumann and Serrano (2008) and Foster and Hart (2009), we assume that our gambles have finite support. We can easily accommodate general distributions, provided they are bounded. Let $\mathcal{B}$ be the collection of all *bounded gambles*: a random variable $g \in \mathcal{B}$ if $\mathbf{E}[g] > 0$, $\mathbf{P}[g < 0] > 0$, and there is a constant B such that $|g| \leq B$. Our results continue to hold on the larger domain $\mathcal{B}$. An easy way to see this is to take, for every g in $\mathcal{B}$, gambles g_1 and g_2 in $\mathcal{G}$ such that $g_1 \leq g \leq g_2$, and $g_2 - g_1$ and $L(g_1) - L(g_2)$ are arbitrarily small (use Monotonicity).

(h) CRRA(γ) *for* $\gamma < 1$. Consider the CRRA(γ) utility function $u(x) = x^{1-\gamma}$ for $0 \leq \gamma < 1$. Can one define its critical wealth $R_\gamma(g)$? Not for all gambles, since for each such u_γ, there are gambles g for which (2) has no solution. The reason is that these gambles are always accepted by these utilities (which are not sufficiently risk averse; cf. Hart 2011). See the discussion following Proposition 10 in Section 6.2 and Lemma 20 in Appendix A.

[26]While the rule that one always takes the proportion $w/K(g) \equiv w/R_2(g)$ of the gamble g at wealth w is the rule that maximizes expected log utility, a similar statement does *not* hold when maximizing a CRRA-γ utility function u_γ with $\gamma \neq 1$. Indeed, for such u_γ, the optimal proportion of g taken at w changes when g may be followed by another gamble h (the additive separability of log is critical here).

Appendices

Here we provide the formal proofs and some additional results: Theorems 1 and 3 are proved in Appendix A, Theorem 4 is proved in Appendix B, and Appendix C deals with the indispensability of the axioms. Finally, Appendix D provides a characterization of those functions Q that satisfy our axioms except Scaling (but with an additional continuity axiom).

Appendix A: Proof of Theorems 1 and 3

We prove here our main result; for an outline of the proof, see Sections 6.1 and 6.2.

The proof is divided into three parts. In Section A.1, we show that the functions R_γ for $\gamma \geq 1$ are well defined (Lemma 5) and strictly increasing in γ (Lemma 8); in Section A.2, that these functions satisfy the four axioms (Proposition 6); and in Section A.3, that these are the only functions to satisfy the four axioms (Proposition 7). Several consequences of the axioms are provided in Section A.3.

Since applying increasing linear transformations to the u_γ functions does not affect (2) and (3), in the proofs, we use the following convenient version of u_γ (see Lemma 13 below):

$$u_\gamma(x) := \begin{cases} \dfrac{1}{1-\gamma}(x^{1-\gamma}-1) & \text{for } 0 \leq \gamma < 1 \\ \log(x) & \text{for } \gamma = 1 \\ \dfrac{1}{\gamma-1}(1-x^{-(\gamma-1)}) & \text{for } \gamma > 1. \end{cases} \tag{6}$$

For every $\gamma \geq 0$, the utility function $u_\gamma : \mathbb{R}_+ \to \mathbb{R}$ has constant relative risk aversion (CRRA) equal to γ, i.e., $-xu''_\gamma(x)/u'_\gamma(x) = \gamma$ for every $x > 0$.

A.1 *The functions R_γ*

In the next lemma, we collect several useful properties of the functions u_γ; we omit the proofs as they are immediate.

Lemma 13. *(i) For each $\gamma \geq 0$, the function u_γ is strictly increasing, $u_\gamma(1) = 0$, and $u'_\gamma(1) = 1$.*

(ii) For each $\gamma > 0$, the function u_γ is strictly concave.

(iii) If $0 \leq \beta < \gamma$, then $u_\beta(x) > u_\gamma(x)$ for every $x \neq 1$, and there exists a strictly increasing and strictly concave function $\psi \equiv \psi_{\beta,\gamma}$ with $\psi(0) = 0$ such that $u_\gamma(x) = \psi(u_\beta(x))$ for every $x > 0$.

(iv) $u_\gamma(x)$ is continuous in γ for every $x > 0$.

With u_γ given by (6), the equality (2) that defines $R_\gamma(g)$ can be rewritten as

$$\mathbf{E}\left[u_\gamma\left(1 + \frac{1}{R_\gamma(g)}g\right)\right] = u_\gamma(1) = 0 \tag{7}$$

(cf. (1) and (3)). We now prove Lemma 5 (that $R_\gamma(g)$ is well defined) and Lemma 8 (that $R_\gamma(g)$ increases in γ) together.

Proof of Lemmata 5 and 8. Fix the gamble $g = (x_i, p_i)_{i=1}^m$. The function $\phi_\gamma(\lambda) := \mathbf{E}[u_\gamma(1 + \lambda g)]$ satisfies the following conditions (see Lemma 13(i) and (ii)): it is strictly concave, $\phi_\gamma(0) = 0$, $\lim_{\lambda\to(1/L(g))^-} \phi_\gamma(\lambda) = -\infty$ (this is where $\gamma \geq 1$ is used), $\phi'_\gamma(\lambda) = \sum_{i=1}^m p_i x_i (1 + \lambda x_i)^{-\gamma}$, and $\phi'_\gamma(0) = \mathbf{E}[g] > 0$; therefore $\phi_\gamma(\lambda)$ has a unique root $\bar\lambda_\gamma$ in the open interval[27] $(0, 1/L(g))$. By (7), $R_\gamma(g) = 1/\bar\lambda_\gamma$, proving Lemma 5.

Moreover, $\beta < \gamma$ implies by Lemma 13(iii) that

$$\phi_\gamma(\bar\lambda_\beta) = \mathbf{E}\big[\psi(u_\beta(1 + \bar\lambda_\beta g))\big] < \psi\big(\mathbf{E}[u_\beta(1 + \bar\lambda_\beta g)]\big) = \psi(0) = 0$$

(the strict inequality follows from the strict concavity of ψ). Now $\phi_\gamma(\lambda) \geq 0$ for every $\lambda \in [0, \bar\lambda_\gamma]$ (since ϕ_γ is concave and $\phi_\gamma(0) = \phi_\gamma(\bar\lambda_\gamma) = 0$); hence, $\bar\lambda_\beta$ cannot lie in this interval and so $\bar\lambda_\beta > \bar\lambda_\gamma$, proving Lemma 8. □

A.2 *The functions R_γ for $\gamma \geq 1$ satisfy the axioms*

In this section, we prove Proposition 6 (that R_γ satisfies the four axioms).

Proof of Proposition 6. Distribution and Scaling are immediate from the definition of R_γ and Lemma 5 (cf. Lemma 10 in Foster and Hart 2009). Monotonicity holds since u_γ is a strictly increasing function (cf. the proof of Proposition 6 in Foster and Hart 2009).

To prove that Compound Gamble holds, take g, h, A, and x to be as in the statement of the axiom; put $r := R_\gamma(g)$, and so $R_\gamma(h) = r + x$. Conditioning on the event A, where $g = x$ and $f = x + h$, we get

$$\begin{aligned} \mathbf{E}[u_\gamma(r + f)|A] &= \mathbf{E}[u_\gamma(r + x + h)|A] = \mathbf{E}[u_\gamma(r + x + h)] \\ &= u_\gamma(r + x) = \mathbf{E}[u_\gamma(r + g)|A] \end{aligned} \tag{8}$$

(the second equality follows from the independence of h and A, and the third follows from (2) since $R_\gamma(h) = r + x$). On the complementary event A^C, where $f = g$, we get

$$\mathbf{E}[u_\gamma(r + f)|A^C] = \mathbf{E}[u_\gamma(r + g)|A^C]. \tag{9}$$

Combining (8) and (9), and then using (2) for g yields

$$\mathbf{E}[u_\gamma(r + f)] = \mathbf{E}[u_\gamma(r + g)] = u_\gamma(r).$$

But this says, again by (2) (and Lemma 5), that $R_\gamma(f) = r = R_\gamma(g)$. □

A.3 *Only the functions R_γ for $\gamma \geq 1$ satisfy the axioms*

We start with three consequences of the axioms, concerning first-order stochastic dominance, the relation to the maximal loss, and multiplicative compounding (see Propositions 14, 9, and 5, respectively). Throughout this section, Q is a fixed wealth-requirement function that satisfies the four axioms of Section 2.1.

[27] Compare Figure 1 and the proof of Lemma 9 in Foster and Hart (2009).

A.3.1 *First-order stochastic dominance* Let $g, h \in \mathcal{G}$. The gamble g *first-order stochastically dominates* the gamble h, which we write as $g \text{ SD}_1 h$, if $\mathbf{P}[g \geq c] \geq \mathbf{P}[h \geq c]$ for every real constant c, with strict inequality for some c; informally, g gets higher values than h. As is well known, $g \text{ SD}_1 h$ if and only if there exist g' and h' defined on the same probability space such that g and g' have the same distribution, h and h' have the same distribution, $g' \geq h'$, and $g' \neq h'$.

Proposition 14. *If $g \text{ SD}_1 h$, then $Q(g) < Q(h)$.*

Proof. Taking g' and h' as above, we have $Q(g) = Q(g') < Q(h') = Q(h)$ by Distribution, Monotonicity, and again Distribution. □

A.3.2 *Maximal loss and wealth requirement* Here we prove Proposition 9: the wealth requirement must always be strictly greater than any possible loss. We first prove a weak inequality.

Lemma 15. *$Q(g) \geq L(g)$ for every g.*

Proof. By way of contradiction, let $h = (x_1, p_1; \ldots; x_m, p_m)$ satisfy $Q(h) < L(h)$. Without loss of generality (use the Scaling axiom), assume that $Q(h) = 1 < L(h)$.

For every $n \geq 2$, let $g_n := (n, 1/2; -1, 1/2) \in \mathcal{G}$ and put $q_n := Q(g_n) > 0$. By Monotonicity, we have $q_n \leq q_2$. Let f_n be the compound gamble

$$\begin{aligned} f_n &:= g_n + \mathbf{1}_{[g_n = n]}(q_n + n)h \\ &= \left(n + (q_n + n)x_1, \tfrac{1}{2}p_1; \ldots; n + (q_n + n)x_m, \tfrac{1}{2}p_m; -1, \tfrac{1}{2}\right). \end{aligned}$$

Since $Q((q_n + n)h) = q_n + n = Q(g_n) + n$ (by Scaling), it follows that $Q(f_n) = Q(g_n) = q_n$ (by Compound Gamble).

Let h' be the gamble

$$h' := \left(x_1 + 1, \tfrac{1}{2}p_1; \ldots; x_m + 1, \tfrac{1}{2}p_m; 0, \tfrac{1}{2}\right)$$

(it is the so-called 1/2-dilution of $h + 1$). Note that h' has negative values since $L(h) > 1$, and is thus a gamble in $\mathcal{G}$. Comparison with f_n shows that $(q_n + n)h'$ first-order stochastically dominates f_n, since $(q_n + n)(x_i + 1) > n + (q_n + n)x_i$ for every $i = 1, \ldots, m$, and $0 > -1$. Therefore, $Q((q_n + n)h') \leq Q(f_n) = q_n$ by Monotonicity (more precisely, Proposition 14) and so

$$Q(h') \leq \frac{q_n}{q_n + n}$$

by Scaling. This holds for each $n \geq 2$, and the limit as $n \to \infty$ yields $Q(h') = 0$ (recall that the q_n's are bounded: $0 < q_n \leq q_2$), contradicting the fact that $Q(h')$ must be positive. □

Remark. Only a weak form of Monotonicity—and, henceforth, of stochastic dominance—was used in the proof of Lemma 15 above, namely, the following condition.

Weak Monotonicity. *If $g \geq h$, then $Q(g) \leq Q(h)$.*

See Appendix C for further discussion and results.

Proof of Proposition 9. By Lemma 15, we have $Q(g) \geq L(g)$. If there is a gamble $h \in \mathcal{G}$ with $Q(h) = L(h)$, then increasing one of the positive values of h yields another gamble $h' \in \mathcal{G}$ with $Q(h') < Q(h)$ (by Monotonicity). But $Q(h) = L(h) = L(h')$ (the first equality is by assumption; the second, by construction), and so $Q(h') < L(h')$, contradicting Lemma 15 for h'. □

A.3.3 *Multiplicative compounding* Given a sequence of random variables $(g_t)_{t=1,2,\ldots}$, a *stopping time* (adapted to the sequence $(g_t)_t$) is a random variable T with values in $\mathbb{N} \cup \{\infty\}$, where $\mathbb{N}$ is the set of positive integers $\{1, 2, \ldots\}$, such that for each finite $n \in \mathbb{N}$, the event $[T = n]$ is determined by $\{g_1, g_2, \ldots, g_n\}$ only; that is, whether one stops after n is determined by the past—the realizations of the first n random variables in the sequence—and not the future. As usual, T is *bounded* if $T \leq N$ for some finite $N < \infty$, and it is *almost surely* (a.s.) *finite* if $\mathbf{P}[T < \infty] = 1$. For every finite $n \in \mathbb{N}$, put $f_n := \prod_{t=1}^{n}(1 + g_t) - 1$. The random variable f_T is then given by $(f_T)(\omega) := f_{T(\omega)}(\omega)$ for each $\omega \in \Omega$, where Ω denotes the probability space over which the g_t—and thus also T—are all defined.

We now prove Proposition 10, which says that the Multiplicative Compounding condition follows from our axioms.

Proof of Proposition 10. Since T is bounded, there is an integer N such that $T \leq N$; we use induction on N. For $N = 1$, we have $T \equiv 1$ and thus $f_T = g_1$. Assume that the result holds for all stopping times $T' \leq N - 1$ and take T such that $T \leq N$. Let $x_1, x_2, \ldots, x_k$ be those values of g_1 where $T > 1$ (recall that the event $[T = 1]$, and so its complement $[T > 1]$, depend only on g_1); assume that $x_1, x_2, \ldots, x_k$ are *distinct*.

For each $j = 1, \ldots, k$, let T_j denote the stopping time T when $g_1 = x_j$; i.e., T_j is determined by the sequence $g_2, g_3, \ldots$ and takes the value that T takes on the sequence $g_1, g_2, g_3, \ldots$ with $g_1 = x_j$ (formally, $T_j(g_2, g_3, \ldots) := T(x_j, g_2, g_3, \ldots)$); thus $2 \leq T_j \leq N$. Put $h_j := \prod_{t=2}^{T_j}(1 + g_t) - 1$. Since the product has $T_j - 1 \leq N - 1$ terms, the induction hypothesis implies that $Q(h_j) = 1$. We have

$$
\begin{aligned}
f_T &= (1 + g_1) + (1 + g_1)\left(\prod_{t=2}^{T}(1 + g_t) - 1\right) - 1 \\
&= g_1 + \sum_{j=1}^{k} \mathbf{1}_{[g_1 = x_j]}(1 + x_j)h_j.
\end{aligned}
$$

Let $f^{(0)} := g_1$, and, for each $j = 1, \ldots, k$, put $A_j := [g_1 = x_j]$ and

$$
f^{(j)} := f^{(j-1)} + \mathbf{1}_{A_j}(1 + x_j)h_j = g_1 + \sum_{\ell=1}^{j} \mathbf{1}_{[g_1 = x_\ell]}(1 + x_\ell)h_\ell.
$$

Now $f^{(j-1)}|_{A_j} = g_1|_{A_j} \equiv x_j$ (here we use the fact that x_j and $x_1, \dots, x_{j-1}$ are distinct), $(1+x_j)h_j$ is independent of A_j (the former depends on g_t for $t \geq 2$ and the latter depends on g_1), and $Q((1+x_j)h_j) = (1+x_j)Q(h_j) = 1+x_j$ (by Scaling). Therefore, by Compound Gamble,[28] if $Q(f^{(j-1)}) = 1$, then $Q(f^{(j)}) = 1$; starting with $Q(f^{(0)}) \equiv Q(g_1) = 1$, after k such applications we get $Q(f^{(k)}) = 1$, completing the proof[29] since $f^{(k)} \equiv f_T$. □

REMARKS. (i) The Monotonicity axiom was not used in the proof of Proposition 10.

(ii) The Compound Gamble axiom is a special case (with $T \leq 2$) of the Multiplicative Compounding condition.

(iii) When $Q(g_t) = q$ for every t, multiplicative compounding yields $Q(f_T) = q$, where $f_T = \prod_{t=1}^{T}(q + g_t) - q$ (take $g'_t = g_t/q$ and use the Scaling axiom).

(iv) A more general version of the Multiplicative Compounding condition is the following:

If $1 + f/q = \prod_{t=1}^{T}(1 + g_t/Q(g_t))$, where $(g_t)_{t=1,2,\dots}$ is a sequence of independent gambles, $q > 0$, and T is a bounded stopping time adapted to the sequence $(g_t)_t$, then $Q(f) = q$.

Interestingly, when Q satisfies Distribution, the above requirement is equivalent to Scaling and Compound Gamble together (indeed, $T = 1$ gives Scaling and $T \leq 2$ gives Compound Gamble).

A.3.4 *Proof of Proposition 7* We are now ready for the main proof here, namely, the proof of Proposition 7: if Q satisfies the four axioms, then Q coincides with one of the R_γ functions for some $\gamma \geq 1$. We start by associating with every gamble g such that $Q(g) = 1$ (and thus, by Proposition 9, $L(g) < 1$) a $\tilde{\gamma}$ such that (essentially) $R_{\tilde{\gamma}}(g) = 1$; this is stated as Lemma 11 in Section 6.2.

LEMMA 16. *For every g with $L(g) < 1$, there exists a unique $\tilde{\gamma} \equiv \tilde{\gamma}(g) > 0$ such that $\mathbf{E}[u_{\tilde{\gamma}}(1+g)] = 0$.*

PROOF. The function $\gamma \to \mathbf{E}[u_\gamma(1+g)]$ is strictly decreasing and continuous (by Lemma 13(iii) and (iv); recall that g has finitely many values). Moreover, $\mathbf{E}[u_0(1+g)] = \mathbf{E}[g] > 0$, and $\mathbf{E}[u_M(1+g)] < 0$ for all $M > 1$ large enough (let $p > 0$ be the probability that g takes the value $L \equiv L(g) \in (0,1)$; then $(M-1)\mathbf{E}[u_M(1+g)] = 1 - \mathbf{E}[(1+g)^{-M+1}] < 1 - p(1-L)^{-M+1} \to_{M\to\infty} -\infty$). □

When $\tilde{\gamma} \geq 1$, this indeed says that $R_{\tilde{\gamma}}(g) = 1$; however, at this point it is conceivable that we have $\tilde{\gamma} < 1$ for some gambles. Next we show that $\tilde{\gamma}$ is not affected by multiplicative compounding (compare Proposition 10).

[28] This is a little more subtle than it may seem at first sight, since, for instance, in the second application of Compound Gamble, h_2 is *not* independent of $f^{(1)} := g_1 + \mathbf{1}_{[g_1=x_1]}(1+x_1)h_1$ (since h_1 and h_2 are not necessarily independent). However, h_2 *is* independent of the event $[f^{(1)} = x_2]$, which is the same as $[g_1 = x_2]$, since x_1 and x_2 are distinct values of g_1.

[29] In particular, $f_T \in \mathcal{G}$.

LEMMA 17. *Let $(g_t)_{t=1,2,\ldots}$ be a sequence of independent gambles with $L(g_t) < 1$ and $\tilde{\gamma}(g_t) = \gamma$ for all t. Let T be a stopping time and let $f_T := \prod_{t=1}^{T}(1 + g_t) - 1$. If either (i) T is bounded or (ii) T is a.s. finite and f_n for $n \le T$ are uniformly bounded, then*[30] $\mathbf{E}[u_\gamma(1 + f_T)] = 0$.

PROOF. When $\gamma \neq 1$, for every integer n, we have $\mathbf{E}[(1 + g_n)^{1-\gamma}] = 1$ (since $\tilde{\gamma}(g_n) = \gamma$), which implies that the sequence $X_n := (1 + f_n)^{1-\gamma} = \prod_{t=1}^{n}(1 + g_t)^{1-\gamma}$ is a martingale:

$$\mathbf{E}[X_n|X_1, X_2, \ldots, X_{n-1}] = X_{n-1}\mathbf{E}[(1 + g_n)^{1-\gamma}] = X_{n-1}$$

(the g_t are independent). Therefore, $\mathbf{E}[X_n] = \mathbf{E}[X_0] = 1$ for every n and so, given the assumptions on the stopping time T, it follows that $\mathbf{E}[X_T] = 1$ or $\mathbf{E}[u_\gamma(1 + f_T)] = 0$.

When $\gamma = 1$, the sequence $X_n := \log(1 + f_n) = \sum_{t=1}^{n}\log(1 + g_t)$ is a martingale (since $\mathbf{E}[X_n] = 0$) and, in the same manner, we get $\mathbf{E}[X_T] = \mathbf{E}[X_n] = \mathbf{E}[X_0] = 0$ or $\mathbf{E}[u_1(1 + f_T)] = 0$. □

Note that for $\gamma \ge 1$, this result follows from Proposition 10 applied to R_γ (recall Proposition 6); however, here we also prove it for $0 < \gamma < 1$.

For every $\beta > 0$ and $z > 1$, let $h_{\beta,z}$ be the two-valued gamble that takes the positive value $z - 1$ and the negative value $1/z - 1$ with appropriate probabilities so that $R_\beta(h_{\beta,z}) = 1$; specifically,

$$h_{\beta,z} = \left(z - 1, 1 - p; \frac{1}{z} - 1, p\right),$$

where

$$p = \frac{1}{z^{\beta-1} + 1}. \tag{10}$$

Clearly $\tilde{\gamma}(h_{\beta,z}) = \beta$ (when $\beta \neq 1$, we have $\mathbf{E}[(1 + h_{\beta,z})^{1-\beta}] = 1$, and when $\beta = 1$, we have $\mathbf{E}[\log(1 + h_{1,z})] = 0$ since then $p = 1/2$ by (10)); and $h \equiv h_{\beta,z}$ is a gamble in $\mathcal{G}$ (since $1/z - 1 < 0$ and $\mathbf{E}[h] = \mathbf{E}[1 + h] - 1 > \mathbf{E}[(1 + h)^{1-\beta}] = 1 - 1 > 0$).

LEMMA 18. *Let $g \in \mathcal{G}$ satisfy $Q(g) = 1$ and $R_\gamma(g) = 1$.*

(i) For each $\beta > \gamma$, there exists z_0 large enough such that $Q(h_{\beta,z}) < 1$ for all $z > z_0$.

(ii) For each $\beta < \gamma$, there exists z_0 large enough such that $Q(h_{\beta,z}) > 1$ for all $z > z_0$.

PROOF. (i) Put $b := \beta - 1$ and $c := \gamma - 1$, and so $b > c$.

We have $1 + g > 0$ (since $L(g) < Q(g) = 1$ by Proposition 9) and so there is $d > 1$ large enough such that $1/d \le 1 + g \le d$. Consider the sequence of gambles $f_n := \prod_{t=1}^{n}(1 + g_t) - 1$,

[30] Unlike case (i), where it is easy to check that $f_T \in \mathcal{G}$ (cf. Proposition 10), in case (ii) the random variable f_T may have infinitely many values and thus need not be in $\mathcal{G}$ (we make use of case (ii) in the proof of Lemma 18).

where the g_t are independent gambles all with the same distribution as g, and define, for $z \geq d$,

$$T := \inf\left\{n : f_n \leq \frac{1}{z} - 1 \text{ or } f_n \geq \frac{z}{d} - 1\right\}.$$

Then T is an a.s. finite stopping time (but in general not bounded).[31] When T is finite, if $1 + f_T \leq 1/z$, then $1 + f_T > 1/(zd)$ (since $1 + f_{T-1} > 1/z$ and $1 + g_T \geq 1/d$), and if $1 + f_T \geq z/d$, then $1 + f_T < z$ (since $1 + f_{T-1} < z/d$ and $1 + g_T \leq d$). Therefore, a.s.,

$$1 + f_T \in \left(\frac{1}{zd}, \frac{1}{z}\right] \cup \left[\frac{z}{d}, z\right). \tag{11}$$

Since T is a.s. finite and f_n for $n \leq T$ are uniformly bounded, $\tilde{\gamma}(g) = \gamma$ implies by Lemma 17(ii) that $\mathbf{E}[u_\gamma(1 + f_T)] = 0$. Letting $q := \mathbf{P}[f_T \leq 1/z - 1] = 1 - \mathbf{P}[f_T \geq z/d - 1]$, when $c = \gamma - 1 \neq 0$, we get from (11) that

$$1 = \mathbf{E}[(1 + f_T)^{-c}] \geq q\left(\frac{1}{zd}\right)^{-c} + (1 - q)\left(\frac{z}{d}\right)^{-c};$$

hence

$$q \geq \frac{D - z^{-c}}{z^c - z^{-c}} \sim \begin{cases} Dz^{-c} & \text{for } c > 0 \\ 1 - Dz^c & \text{for } c < 0, \end{cases} \tag{12}$$

where $D := d^{-c}$, and we write "$A \sim B$" for "$A/B \to 1$ as $z \to \infty$." Similarly, when $c = \gamma - 1 = 0$, we get

$$0 = \mathbf{E}[\log(1 + f_T)] \geq q \log\left(\frac{1}{zd}\right) + (1 - q) \log\left(\frac{z}{d}\right);$$

hence

$$q \geq \frac{\log z - \log d}{2 \log z} \sim \frac{1}{2} \quad \text{for } c = 0. \tag{13}$$

From (10) we get

$$p \sim \begin{cases} z^{-b} & \text{for } b > 0 \\ \frac{1}{2} & \text{for } b = 0 \\ 1 - z^b & \text{for } b < 0. \end{cases}$$

Comparing this with (12) and (13) and recalling that $b > c$ shows that $q > p$ for all z large enough, say $z > z_0$.

Writing $T \wedge n$ for $\min\{T, n\}$, we have $\mathbf{P}[f_{T\wedge n} \leq 1/z - 1] \to_{n\to\infty} \mathbf{P}[f_T \leq 1/z - 1] = q > p$, and so for each $z > z_0$, there is N large enough such that $\mathbf{P}[f_{T\wedge N} \leq 1/z - 1] > p$. This implies that $h_{\beta,z}$ first-order stochastically dominates $f_{T\wedge N}$, since $\mathbf{P}[f_{T\wedge N} \leq z - 1] = 1 = \mathbf{P}[h_{\beta,z} \leq z - 1]$ and $\mathbf{P}[f_{T\wedge N} \leq 1/z - 1] > p = \mathbf{P}[h_{\beta,z} \leq 1/z - 1]$ (recall that $h_{\beta,z}$ has only

[31] As usual, the infimum of an empty set is ∞. The fact that $T = \infty$ has probability 0 is standard: let $x > 0$ be a value of g and let $p > 0$ be its probability; then for K large enough so that $(1 + x)^K \geq z^2/d$ holds, the sequence $(1 + f_{N+n})_{n\geq 1}$ starting with $1 + f_N \in (1/z, z/d)$ reaches the upper bound z/d in K steps with a probability of at least $p^K > 0$.

two values, $z-1$ and $1/z-1$). Now $Q(f_{T\wedge N})=1$ (by Proposition 10), and so $Q(h_{\beta,z})<1$ (by Proposition 14).

(ii) The proof here is similar: letting $T:=\inf\{n: f_n\leq d/z-1 \text{ or } f_n\geq z-1\}$, one shows that $\mathbf{P}[f_{T\wedge N}\geq z-1]>\mathbf{P}[h_{\beta,z}\geq z-1]$ for all z and N large enough. □

LEMMA 19. *There exists $\gamma^*>0$ such that $\tilde{\gamma}(g)=\gamma^*$ for every g with $Q(g)=1$.*

PROOF. If g and g' are two gambles with $Q(g)=Q(g')=1$ but, say, $\tilde{\gamma}(g)<\tilde{\gamma}(g')$, then take β such that $\tilde{\gamma}(g)<\beta<\tilde{\gamma}(g')$. Lemma 18 implies that for all z large enough, we have $Q(h_{\beta,z})<1$ (since $\beta>\tilde{\gamma}(g)$) and also $Q(h_{\beta,z})>1$ (since $\beta<\tilde{\gamma}(g')$), a contradiction. □

LEMMA 20. $\gamma^*\geq 1$.

PROOF. Assume that $0<\gamma^*<1$. Let $z>1/(1-\gamma^*)$ and consider the gamble $g=(2^z-1, 1/2; -1, 1/2)\in\mathcal{G}$. Let $q:=Q(g)$; then $q\geq L(g)=1$ by Lemma 15 and $Q((1/q)g)=1$ by the Scaling axiom, and so $\mathbf{E}[u_{\gamma^*}(1+(1/q)g)]=0$ by Lemma 19. Put $\phi(\lambda):=\mathbf{E}[u_{\gamma^*}(1+\lambda g)]$; then ϕ is a concave function with $\phi(0)=0$ and

$$\phi(1)=\mathbf{E}[u_{\gamma^*}(1+g)]=\frac{1}{1-\gamma^*}\left(\frac{1}{2}(2^z)^{1-\gamma^*}+\frac{1}{2}(0)^{1-\gamma^*}-1\right)>0.$$

Therefore, $\phi(\lambda)>0$ for every $0<\lambda\leq 1$, contradicting $\phi(1/q)=0$ (recall that $q\geq 1$). □

COROLLARY 21. *$Q(g)=R_{\gamma^*}(g)$ for every $g\in\mathcal{G}$.*

PROOF. Let $q:=Q(g)$ and $h:=(1/q)g$. Then $Q(h)=1$ (by Scaling), and so $\mathbf{E}[u_{\gamma^*}(1+(1/q)g)]=0$ (by Lemma 19), which says that $R_{\gamma^*}(g)=q$ (by Lemma 20). □

PROOF OF PROPOSITION 7. This follows from Corollary 21 and Lemma 20. □

A.4 *Conclusion*

PROOF OF THEOREM 3. This follows from Lemma 5 and Propositions 6 and 7. □

PROOF OF THEOREM 1. This follows from Theorem 3 and Lemma 8. □

APPENDIX B: PROOF OF THEOREM 4

In this Appendix, we prove Theorem 4 that characterizes the Aumann–Serrano index. We start with the counterpart of Proposition 10, using the Wealth-Independent Compound Gamble axiom instead of the Compound Gamble axiom.

PROPOSITION 22. *Let Q satisfy the Wealth-Independent Compound Gamble axiom and let $(g_t)_{t=1,2,\ldots}$ be a sequence of independent gambles with $Q(g_t)=1$ for all t. Let T be a bounded stopping time and put*

$$f_T:=\sum_{t=1}^{T}g_t.$$

Then $f_T\in\mathcal{G}$ and $Q(f_T)=1$.

It is noteworthy how the two "compound gamble" axioms lead to multiplicative compounding and additive compounding, respectively. The proof of Proposition 22 is similar to that of Proposition 10 and is omitted.[32]

Proof of Theorem 4. The proof is similar to the proofs in Appendix A, and in some places is simpler; we are thus brief here.

First, note that Q satisfies the axioms if and only if cQ satisfies the axioms for any $c > 0$. To see that R^{AS} satisfies the Wealth-Independent Compound Gamble axiom (the other axioms are immediate), take g, h, A, and f as in the statement of the axiom. Put $\lambda := 1/R^{\mathrm{AS}}(g) = 1/R^{\mathrm{AS}}(h)$. Then $\mathbf{E}[\exp(-\lambda g)] = \mathbf{E}[\exp(-\lambda h)] = 1$. On A, we have $f = x + h$ and so

$$\begin{aligned}\mathbf{E}[\exp(-\lambda f)|A] &= \mathbf{E}[\exp(-\lambda x - \lambda h)] \\ &= \exp(-\lambda x) \cdot \mathbf{E}[\exp(-\lambda h)] = \exp(-\lambda x) \\ &= \mathbf{E}[\exp(-\lambda g)|A].\end{aligned}$$

On A^C we have $f = g$, and so

$$\mathbf{E}[\exp(-\lambda f)|A^C] = \mathbf{E}[\exp(-\lambda g)|A^C].$$

Altogether,

$$\mathbf{E}[\exp(-\lambda f)] = \mathbf{E}[\exp(-\lambda g)] = 1,$$

and so $R^{\mathrm{AS}}(f) = 1/\lambda = R^{\mathrm{AS}}(g)$.

Conversely, let Q satisfy the four axioms. Take $g \in \mathcal{G}$ with $Q(g) = 1$, and put $c := 1/R^{\mathrm{AS}}(g) > 0$. Given a sequence g_t of independent gambles, all with the same distribution as g, let $f_n := \sum_{t=1}^{n} g_t$. Then $\exp(-cf_n)$ is a martingale (since $\mathbf{E}[\exp(-cg_t)] = \mathbf{E}[\exp(-cg)] = 1$) and so $\mathbf{E}[\exp(-cf_n)] = 1$ for every $n \geq 1$.

Next, for $b > 0$ and $z > 1$, let $h_{b,z} \in \mathcal{G}$ take the values $\log z$ and $-\log z$ with probabilities $1 - p$ and p, respectively, where $p = 1/(z^b + 1)$. Then $\mathbf{E}[\exp(-bh_{b,z})] = 1$, and so $1/R^{\mathrm{AS}}(h_{b,z}) = b$. We claim that if $b > c$, then $Q(h_{b,z}) < 1$ for all z large enough, and if $b < c$, then $Q(h_{b,z}) > 1$ for all z large enough. The proof, similar to that of Lemma 18, is as follows: let $d := \max|g| > 0$, and take $z > e^d$. When $b > c$, put $T := \inf\{n : f_n \leq -\log z \text{ or } f_n \geq \log z - d\}$. Then $f_T \in (-\log z - d, -\log z] \cup [\log z - d, \log z)$, and also $\mathbf{E}[\exp(-cf_T)] = 1$ (since $\exp(-cf_n)$ is a martingale and f_T is bounded). This implies that $q := \mathbf{P}[f_T \leq -\log z] \geq (D - z^{-c})/(z^c - z^{-c}) \sim Dz^{-c}$ as $z \to \infty$, where $D := \exp(-cd)$. Comparing this with $p \sim z^{-b}$ shows that there exists z_0 such that $q > p$ for all $z > z_0$; hence for all N large enough, $h_{b,z}$ first-order stochastically dominates $f_{T \wedge N}$. Monotonicity (Proposition 14) implies that $Q(h_{b,z}) < Q(f_{T\wedge N}) = 1$, where we use Proposition 22 for the bounded stopping time $T \wedge N$. The case $b < c$ is similar, with $T := \inf\{n : f_n \leq -\log z + d \text{ or } f_n \geq \log z\}$.

[32] Scaling is no longer needed here. Also, $Q(g_t) = 1$ for all t may be replaced by $Q(g_t) = q$ for all t, with the conclusion that $Q(f_T) = q$.

Therefore, if there is another gamble $g' \in \mathcal{G}$ with $Q(g') = 1$ but $1/R^{\text{AS}}(g') = c' \neq c$, say $c < c'$, then take b such that $c < b < c'$. For large enough z, we get $Q(h_{b,z}) < 1$ (since $b > c$) and also $Q(h_{b,z}) > 1$ (since $b < c'$), a contradiction. This shows that there exists a constant $c^* > 0$ such that $Q(g) = 1$ implies that $R^{\text{AS}}(g) = 1/c^*$, or $Q(g) = c^* R^{\text{AS}}(g)$ for any g with $Q(g) = 1$. The Scaling axiom applied to Q and R^{AS} implies that $Q(g) = c^* R^{\text{AS}}(g)$ for any $g \in \mathcal{G}$. □

Remarks. (i) Since $\alpha := 1/R^{\text{AS}}(g)$ (the "adjustment coefficient" of g) satisfies $\mathbf{E}[\exp(-\alpha g)] = 1$, the Additive Compounding condition (and its special case, the Wealth-Independent Compound Gamble axiom) immediately follows (for instance, $\exp(-\alpha \sum_{t=1}^{N} g_t)$ is a martingale when the g_t are i.i.d. and distributed like g). Interestingly, Theorem 4 shows that in the presence of the other axioms, this is the *only* way to get the Additive Compounding condition.

(ii) The Scaling axiom is used only in the last line of the proof. Without Scaling, we get the conclusion that Q is ordinally equivalent to R^{AS}, i.e., Q is a monotonic transformation of R^{AS}. (Indeed, we have shown that there exists a constant $c^* > 0$ such that $Q(g) = 1$ implies that $R^{\text{AS}}(g) = 1/c^*$; in the same way, one shows that for every $q > 0$, there exists $c^*(q) > 0$ such that $Q(g) = q$ implies that $R^{\text{AS}}(g) = 1/c^*(q)$.)

Appendix C: The axioms are indispensable

We show here that the axioms of Section 2.1—Monotonicity, Scaling, and Compound Gamble—are *indispensable*: for each one we provide a function Q with $Q(g) < \mathbf{R}(g)$ for some g (or even all g) in $\mathcal{G}$ that satisfies all the other axioms. We conclude with a discussion of the Distribution axiom.

The Monotonicity axiom

The function L that associates to each gamble g its maximal loss $L(g)$ clearly satisfies Distribution and Scaling. To see that L also satisfies Compound Gamble, take g, h, A, and f as in the statement of the axiom; thus $g|_A \equiv x$ and $L(h) = L(g) + x$. Then $\min f|_A = \min(x + h) = x - L(h) = -L(g)$ and $\min f|_{A^c} = \min g|_{A^c} \geq \min g = -L(g)$, and so $\min f = -L(g)$, i.e., $L(f) = L(g)$. But L does not satisfy Monotonicity: just increase any of the outcomes of g that are above $L(g)$ and then L does not decrease. Of course, $L(g) < \mathbf{R}(g)$ for all $g \in \mathcal{G}$.

Clearly, L satisfies the Weak Monotonicity axiom, with $Q(g) \leq Q(h)$ instead of $Q(g) < Q(h)$; see the Remark following the proof of Lemma 15. Interestingly, replacing the Monotonicity axiom with this weaker axiom yields the following result (compare our main result Theorem 1).

Proposition 23. *The* minimal *function that satisfies the four axioms Distribution, Scaling, Weak Monotonicity, and Compound Gamble is the maximal loss function L.*

Proof. We have just seen that L satisfies these axioms. Lemma 15 (see the Remark following its proof) shows that $Q \geq L$ for any Q satisfying these four axioms, so L is indeed the minimal function. □

The Scaling axiom

Take the utility function $u(x) = x + \log(x)$ for all $x > 0$ (note that it is *not* a CRRA utility), and define $Q(g) := r$ as the unique solution $r > L(g)$ of the equation $\mathbf{E}[u(r+g)] = u(r)$.

We claim that Q is well defined: for every $x > 0$, we have $u'(x) > 0$, $u''(x) < 0$, $\rho(x) \equiv -u''(x)/u'(x) > 0$, $\rho'(x) < 0$; thus u is a Decreasing Absolute Risk Aversion (DARA) utility function. An agent with such a utility is monotonic in his decisions: for each g, there is a unique critical wealth level $w_0 \equiv w_0(g)$ with $0 \leq w_0 \leq \infty$ such that at all wealth levels $w < w_0$, the agent rejects g (i.e., $\mathbf{E}[u(w+g)] < u(w)$), and at all wealth levels $w > w_0$, it accepts g (i.e., $\mathbf{E}[u(w+g)] > u(w)$); see Pratt (1964), Yaari (1969), Dybvig and Lippman (1983), Hart (2011). Put $\phi(r) := \mathbf{E}[u(r+g)] - u(r)$ (for our specific u); then $\lim_{r\to L(g)^+} \phi(r) = -\infty$ (since $\lim_{x\to 0^+} u(x) = -\infty$), and $\phi(\mathbf{R}(g)) = \mathbf{E}[g] > 0$ (since $\mathbf{E}[\log(\mathbf{R}(g) + g)] = \log(\mathbf{R}(g))$). Therefore, the equation $\phi(r) = 0$ has a solution r in the interval $(L(g), \mathbf{R}(g))$, which is unique since that is precisely the critical w_0.

Moreover, Q satisfies Distribution, Monotonicity (if $g \geq h$ and $g \neq h$, then $\mathbf{E}[u(Q(h)+g)] > \mathbf{E}[u(Q(h)+h)] = u(Q(h))$ since u is strictly increasing; recalling the function ϕ of the previous paragraph, $Q(h)$ thus lies in the region where ϕ is positive, and so it is larger than the root $Q(g)$ of ϕ), and Compound Gamble (the proof of Proposition 6 did not use any special property of the u_γ function beyond the existence and uniqueness of the solution of (2), and so it applies to our u too). However, Q does not satisfy Scaling, since u is *not* homogeneous of degree 1; e.g., for $g = (2, 1/2; -1, 1/2)$, we get $Q(g) \approx 1.155$ and $Q(2g) \approx 2.098 \neq 2Q(g)$.

As we have seen above, this function Q satisfies $Q(g) < \mathbf{R}(g)$ for every g.

This example suggests that without Scaling, the other axioms may characterize the critical wealth functions for a wider class of utilities than CRRA. In Appendix D we show that is indeed the case, once we add a Continuity axiom (which is needed to provide a certain degree of regularity; in the results of our Theorems 1 and 3, regularity is implicitly provided by Scaling).

The Compound Gamble axiom

Take $Q(g) = \delta\mathbf{R}(g) < \mathbf{R}(g)$ for some fixed $0 < \delta < 1$: it satisfies all the axioms (as well as the δ-Compound Gamble axiom; see Section 7(f)) except the Compound Gamble axiom.[33]

The Distribution axiom

The Distribution axiom is needed since it is more convenient to work with random variables than with probability distribution. Without it, one can easily get a counterexample to the result of Theorem 3: fix a probability space Ω_0, and put $Q(g) = R_2(g)$ for all g defined on Ω_0 and $Q(g) = R_1(g)$ otherwise. However, we conjecture that one cannot go beyond this: a function Q that satisfies the other three axioms must equal on each

[33]Another example that satisfies the first three axioms but not the Compound Gamble axiom is R^{AS} (see Theorem 4); while R^{AS} differs from all the R_γ of Theorem 3, it is not always less than $\mathbf{R}$.

probability space one of the R_γ functions (and thus the result of the Main Theorem may well hold without the Distribution axiom).[34]

Appendix D: Without Scaling

In the previous appendix, we saw a critical wealth function that does not satisfy the Scaling axiom: it is obtained from a non-CRRA utility. This suggests[35] that one may characterize the class of functions Q that satisfy all axioms except Scaling. We show here that the other axioms, together with a certain Continuity axiom,[36] characterize the critical wealth functions for a class of DARA (Decreasing Absolute Risk Aversion) utilities. Moreover, we show that the infimum of all these functions Q is the maximal loss function L. The proofs here are quite different from those of our main results.

The continuity requirement is stated as follows.

Continuity. *Let g_n be a sequence of gambles in $\mathcal{G}$, all with the same finite support S, such that $g_n \to g$ in distribution*[37] *(note that g need not be a gamble in $\mathcal{G}$).*

(i) If g is a gamble (i.e., $g \in \mathcal{G}$) and its support is also[38] *S, then $Q(g_n) \to Q(g)$.*

(ii) If $\mathbf{E}[g] = 0$ and g is not identically 0, then $Q(g_n) \to \infty$.

(iii) If $g \geq 0$ and g is not identically 0, then $Q(g_n) \to L(S)$, where $L(S) := -\min S$ (which equals $L(g_n)$ for all n).

An immediate consequence of Continuity is the next lemma.

Lemma 24. *Let $a, b > 0$. For every $c > b$, there exists q such that $g_{a,b;q} := (a, q; -b, 1-q) \in \mathcal{G}$ and $Q(g_{a,b;q}) = c$.*

Proof. The function $\theta(q) := Q(g_{a,b;q})$ satisfies the following conditions: it is continuous for $0 < q < b/(a+b)$ (by (i)); $\theta(q) \to 0$ when $q \to b/(a+b)$ (by (ii)); and $\theta(q) \to b$ when $q \to 0$ (by (iii)). □

Let $\mathcal{U}_0$ be the set of utility functions $u : \mathbb{R}_+ \to \mathbb{R}$ that are increasing, concave, continuously differentiable (i.e., C^1), and satisfy the following conditions[39]: $\lim_{x\to 0^+} u(x) = -\infty$; the absolute risk-aversion coefficient $\rho_u(x) \equiv -u''(x)/u'(x)$ (which exists almost everywhere) is decreasing; and $\lim_{x\to\infty} \rho_u(x) = 0$. Our characterization result is the following proposition.

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[34]The proof, based on arguments of the kind used in Section A.3, may be quite complex.

[35]We thank the referees for this suggestion.

[36]Continuity is needed to provide a certain degree of "regularity" for Q; in the results of our Theorems 1 and 3, this regularity is implicitly provided by Scaling. In addition, we can no longer prove that $Q(g) > L(g)$ for all g (cf. Proposition 9), and so it becomes part of Continuity.

[37]The support S of all g_n is fixed and finite, say of size m, so convergence in distribution is the same as convergence of the vectors of probabilities (as points in the unit simplex of $\mathbb{R}^m$).

[38]This means that there is no value in S whose probability vanishes in the limit g.

[39]See Hart (2011) for a discussion of the first two conditions ($u(0^+) = -\infty$ and DARA). Note that $u \in \mathcal{U}_0$ need not be twice continuously differentiable (i.e., C^2): take, for instance, $u(x) = 1 - 1/x$ for $x \leq 1$ and $u(x) = \log(x)$ for $x \geq 1$ (thus $\rho_u(x) = 2$ for $x < 1$ and $\rho_u(x) = 1$ for $x > 1$).

Proposition 25. *The function Q satisfies the Distribution, Monotonicity, Compound Gamble, and Continuity axioms if and only if there exists a utility function $u \in \mathcal{U}_0$ such that for every $g \in \mathcal{G}$,*

$$Q(g) = w \quad \textit{if and only if} \quad \mathbf{E}[u(w+g)] = u(w). \tag{14}$$

The proof is split into two parts: Lemma 26 and Lemma 27.

Lemma 26. *Let $u \in \mathcal{U}_0$ and let Q be given by (14). Then Q is well defined and satisfies the four axioms of Proposition 25.*

Proof. First, we claim that for every $g \in \mathcal{G}$ and $u \in \mathcal{U}_0$ the equation

$$\mathbf{E}[u(w+g)] = u(w) \tag{15}$$

has a unique solution $w > L(g)$. Indeed, for every $d > 0$, we have $\rho_u(x) > \rho_u(x+d)$ for all x, and so $u(x) = \psi(u(x+d))$ with ψ strictly increasing and strictly concave. From this it follows that $\mathbf{E}[u(w+d+g)] \le u(w+d)$ implies $\mathbf{E}[u(w+g)] < u(w)$ (using the same arguments as in the proof of Proposition 2 in[40] Hart 2011, with $u_1(x) = u(x)$ and $u_2(x) = u(x+d)$). Therefore, there can be at most one $w = w_0$ satisfying (15), and $\mathbf{E}[u(w+g)] < u(w)$ for all $w < w_0$, and $\mathbf{E}[u(w+g)] > u(w)$ for all $w > w_0$. To see that such w_0 exists, note that $\mathbf{E}[u(w+g)] < u(w)$ as w decreases to $L(g)$ (the left-hand side goes to $-\infty$) and $\mathbf{E}[u(w+g)] > u(w)$ as w goes to ∞ (cf. Proposition 4(iv) in Hart 2011). In summary, $Q(g) = w_0$ is well defined and, moreover,

$$\operatorname{sign}\big(\mathbf{E}[u(w+g)] - u(w)\big) = \operatorname{sign}(w - Q(g)). \tag{16}$$

Second, we show that the axioms are satisfied. The Distribution axiom is immediate. For Monotonicity, note that $g' \gneqq g$ implies $\mathbf{E}[u(Q(g)+g')] > \mathbf{E}[u(Q(g)+g)] = u(Q(g))$, and so $Q(g') < Q(g)$ by (16). For Compound Gamble, use the proof of Proposition 6. For Continuity, (i) is immediate; for (ii), if $Q(g_n)$ converges to some finite $w \ge L(S) > 0$, then $\mathbf{E}[u(w+g)] = u(w)$, contradicting the strict concavity of u (recall that $\rho(x) > 0$ and thus $u''(x) > 0$ for a.e. x) since $\mathbf{E}[g] = 0$ and g is not identically 0; for (iii), if $Q(g_n)$ converges to some $w > L(S)$, then $\mathbf{E}[u(w+g)] = u(w)$, contradicting the strict monotonicity of u, since $g \gneqq 0$. □

Lemma 27. *Let Q satisfy the four axioms of Proposition 25. Then there exists $u \in \mathcal{U}_0$ such that Q is given by (14).*

We first prove a simpler result.

Lemma 28. *Let Q satisfy the four axioms of Proposition 25 and let $0 < s < t$. Then there exists a strictly increasing function $u : [s,t] \to \mathbb{R}$ such that $\mathbf{E}[u(Q(g)+g)] = u(Q(g))$ holds for all $g \in \mathcal{G}$ with $Q(g) + g \in [s,t]$. Moreover, u is unique up to positive affine transformations.*

[40] It can be verified that the proofs in Hart (2011) apply also when the nonincreasing function $\rho_u(x)$ is defined only a.e., since $\rho_u(x)$ is the derivative of the concave function $-\log u'(x)$.

Proof. Put $u(s) := 0$ and $u(t) := 1$. For every w in the open interval (s,t), put $u(w) := q_w$, where q_w is given by Lemma 24 for $a = t - w$ and $b = w - s$, i.e., so that $Q(g_{t-w,s-w;q_w}) = w$. Since $Q(g_{t-w,s-w;q})$ is strictly decreasing in q and in w by Monotonicity, it follows that $q_w = u(w)$ is unique and strictly increasing in w.

To show that $\mathbf{E}[u(Q(g)+g)] = u(Q(g))$, consider first g such that $Q(g) + g \in \{s,t\}$. Thus $g = (t-w, p; s-w, 1-p)$ for some p, where $w := Q(g)$. But then $p = q_w$ by the uniqueness of q_w, and so $\mathbf{E}[u(Q(g)+g)] = q_w u(w+t-w) + (1-q_w)u(w+s-w) = q_w = u(w) = u(Q(g))$ (recall that $u(t) = 1$ and $u(s) = 0$).

Next, take any g such that $Q(g) + g \in (s,t)$, say $g = (x_1, p_1; \dots; x_m, p_m)$, and let $w := Q(g)$. For each i, let h_i be a two-point gamble $h_i = (t - w - x_i, r_i; s - w - x_i, 1 - r_i)$, where $r_i := u(w + x_i)$, or, equivalently, $Q(h_i) = w + x_i$; assume that g and all h_i are independent. Then, by Compound Gamble, $f := g + \sum_i \mathbf{1}_{g=x_i} h_i$ satisfies $Q(f) = Q(g) = w$. But f has only two values, $t - w$ and $s - w$, and so by the argument in the previous paragraph, $\mathbf{E}[u(w+f)] = u(w)$. Now $\mathbf{E}[u(w+g)] = \sum_i p_i u(w+x_i) = \sum_i p_i \mathbf{P}[h_i = t - w - x_i] = \sum_i p_i \mathbf{P}[f = t - w | g = x_i] = \mathbf{P}[f = t - w] = \mathbf{E}[u(w+f)] = u(w)$. Finally, if g takes values x_i such that $Q(g) + x_i$ equals either s or t, then we need no further gamble h_i after such an x_i.

The uniqueness of u up to positive affine transformations follows by considering again the two-point gambles $g_{t-w,s-w;q_w}$, which yield $u(w) = q_w u(t) + (1-q_w)u(s)$. □

Proof of Lemma 27. For each integer $n \geq 2$, let u_n be given by Lemma 28 for $s = 1/n$ and $t = n$, normalized by $u_n(1) = 1$ and $u_n(2) = 2$. For $n' > n$, the function $u_{n'}$ restricted to $[1/n, n]$ must be a positive affine transformation of u_n; since u_n and $u_{n'}$ coincide at 1 and 2, they coincide on all of $[1/n, n]$. Thus $u_{n'}$ is an extension of u_n, and so the function $u : \mathbb{R}_+ \to \mathbb{R}$ defined by $u(w) := u_n(w)$ for any n such that $w \in [1/n, n]$ satisfies $\mathbf{E}[u(Q(g) + g)] = u(Q(g))$ for every g (just take n large enough so that $Q(g) + g \in [1/n, n]$). Thus $\mathbf{E}[u(w+g)] = u(w)$ when $w = Q(g)$, for all $g \in \mathcal{G}$.

Conversely, let $w := Q(g)$ and assume that $\mathbf{E}[u(w' + g)] = u(w')$ for some $w' \neq w$ (where $w, w' > L(g)$). Consider first the case $w' > w$. By moving probability mass from the positive values of g to its lowest value $-L(g)$, we get g' that satisfies $Q(g') = w'$ and has the same support as g (such a g' exists since the intermediate value $w < w' < \infty$ is attained by Continuity). Therefore, $\mathbf{E}[u(w' + g')] = \mathbf{E}[u(w' + g)]$ (they both equal $u(w')$: the first by the previous paragraph for g' and the second by our assumption), which contradicts the construction of g' as u is strictly increasing and thus $u(w' + x) > u(w' - L(g))$ for every $x > 0$. The case $w' < w$ is handled in a similar manner, moving probability mass from the negative values of g to its highest value $M(g)$.

Thus u is an increasing function, and the equation $\mathbf{E}[u(w+g)] = u(w)$ has a unique solution $w > L(g)$ for every $g \in \mathcal{G}$.

To show that u is a concave function, let $0 < \delta < x$. By Lemma 24, there is $1/2 < p < 1$ such that $Q(g_{\delta,-\delta;p}) = x$, and so $pu(x+\delta) + (1-p)u(x-\delta) = u(x)$. Since $p > 1/2$ and u is increasing, it follows that $(1/2)u(x+\delta) + (1/2)u(x-\delta) < u(x)$, or $u(x+\delta) - u(x) < u(x) - u(x-\delta)$, which implies, in particular, that u is concave on the δ-grid (i.e., when restricted to $x = \delta, 2\delta, \dots, n\delta, \dots$). Since this holds for all $\delta > 0$, it follows that u is concave.

Next, we claim that for every $\varepsilon > 0$ and $\varepsilon < s < t$, we have

$$\frac{u(s+\varepsilon)-u(s)}{u(t+\varepsilon)-u(t)} < \frac{u(s)-u(s-\varepsilon)}{u(t)-u(t-\varepsilon)}. \tag{17}$$

Indeed, let p, p' be such that $Q(g_{\varepsilon,-\varepsilon;p}) = s$ and $Q(g_{\varepsilon,-\varepsilon;p'}) = t$, and so $s < t$ implies $p > p'$ by Monotonicity. From (14), we get

$$\frac{u(t)-u(t-\varepsilon)}{u(t+\varepsilon)-u(t)} = \frac{p'}{1-p'} < \frac{p}{1-p} = \frac{u(s)-u(s-\varepsilon)}{u(s+\varepsilon)-u(s)},$$

which yields (17).

Let $v(x) := \log u'_+(x)$, where $u'_+(x) = \lim_{\varepsilon\to 0^+}(u(x+\varepsilon)-u(x))/\varepsilon$ is the right-derivative of u at x. We claim that v is a convex function: for every $\delta > 0$ and $0 < x < y$, we have $v(x+\delta)-v(x) \le v(y+\delta)-v(y)$ or, equivalently, $u'_+(x+\delta)/u'_+(x) \le u'_+(y+\delta)/u'_+(y)$. Indeed, otherwise we would have for all small enough $\varepsilon > 0$,

$$\frac{u(x+\delta+\varepsilon)-u(x+\delta)}{u(x+\varepsilon)-u(x)} > \frac{u(y+\delta+\varepsilon)-u(y+\delta)}{u(y+\varepsilon)-u(y)}. \tag{18}$$

Taking a large enough integer m so that (18) holds for $\varepsilon = \delta/m$, and then using (17) with $s = x+i\varepsilon$ and $t = x+i\varepsilon$ for $i = m, m-1, \ldots, 1$ yields the chain of inequalities

$$\begin{aligned}\frac{u(x+(m+1)\varepsilon)-u(x+m\varepsilon)}{u(y+(m+1)\varepsilon)-u(y+m\varepsilon)} &< \frac{u(x+m\varepsilon)-u(x+(m-1)\varepsilon)}{u(y+m\varepsilon)-u(y+(m-1)\varepsilon)} \\ &< \cdots < \frac{u(x+\varepsilon)-u(x)}{u(y+\varepsilon)-u(y)},\end{aligned}$$

which contradicts (18). Thus v is a convex function as claimed. Therefore, v is continuous and so u'_+ is continuous; hence u' exists everywhere and is continuous (since $u'_-(x) = \lim_{y\to x^-} u'_+(y)$), i.e., $u \in C^1$. Moreover, $-v'(x) = -u''(x)/u'(x) = \rho(x)$ exists at all points x except at most countably many, and so the absolute risk-aversion coefficient of u is a nonincreasing function.

If $\lim_{x\to 0^+} u(x) > -\infty$, then there exist gambles g such that $\mathbf{E}[u(w+g)] > u(w)$ for all $w > L(g)$, and thus (15) has no solution for such g (see Proposition 6(ii) in Hart 2011; take for instance $g = (1, 1-\varepsilon; -1, \varepsilon)$ for small enough $\varepsilon > 0$). Similarly, if $\lim_{x\to\infty} \rho(x) > 0$, then there exist gambles g such that $\mathbf{E}[u(w+g)] < u(w)$ for all $w > L(g)$, and thus (15) has no solution for such g (see Proposition 4(i) in Hart 2011; take g with $R^{\mathrm{AS}}(g) < 1/\lim_{x\to\infty} \rho(x)$). Finally, $\rho(x)$ must be strictly decreasing, since otherwise there is an interval where ρ is constant, and then there exist gambles for which (15) has multiple solutions (if $\rho(x) = \alpha$ for all x in some interval, then u is CARA-α in that interval; take g with small values and $R^{\mathrm{AS}}(g) = 1/\alpha$).

Thus $u \in \mathcal{U}_0$. □

For $u \in \mathcal{U}_0$, let $Q_u : \mathcal{G} \to \mathbb{R}_+$ denote the function given by (14).

Proposition 29. *For every $g \in \mathcal{G}$,*

$$\inf_{u\in\mathcal{U}_0} Q_u(g) = L(g).$$

Proof. For every $m > 0$, take $u_m(x) = x + (1/m)\log x$. It is easy to see that $u_m \in \mathcal{U}_0$, and that (14) becomes $\mathbf{E}[\log(1 + g/Q_{u_m}(g))] = -m\mathbf{E}[g]$. As $m \to \infty$, the right-hand side converges to $-\infty$, and therefore, $Q_{u_m}(g) \to L(g)$ (see the proof of Lemma 1 and Figure A1 in Appendix A of Foster and Hart 2009). □

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