Collaboration between and within groups*

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June 4, 2026

Abstract

We study the ability of multi-group teams to undertake binary projects in a decentralized environment. The equilibrium outcomes of our model display familiar features in collaborative settings, including inefficient gradualism, inaction, and contribution cycles, wherein groups alternate taking responsibility for moving the project forward. Expected delay grows more than proportionally with project size, and some welfare-enhancing projects are not completed, even as agents become arbitrarily patient. A team composed of two equally large groups can complete larger projects than a fully homogeneous team, even as the difference in preferences for completion among the two groups is arbitrarily small. Moreover, if the project is sufficiently large, the two-group team always completes the project strictly faster.

1 Introduction

In the textbook notion of a team, homogeneous agents work together to attain a common goal. In many real-life scenarios, however, teams are composed of distinct

*Iaryczower gratefully acknowledges financial support from NSF grant SBE 1757191. Oliveros gratefully acknowledges the support of the Economic and Social Research Council (UK), ES/S01053X/1.

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groups. In these contexts, agents collaborate with both peers and "outsiders", who can have a different valuation for completion of the project. In a research lab, for instance, senior scholars collaborate with junior researchers, for whom a publication may have a greater career impact. In corporations, new product launches usually involve both Product and Operations personnel, who jointly develop new infrastructure to make the product available to clients. In open source software development, both amateur and professional developers write the code that drives projects like Linux. Collaboration between groups is also paramount in political economy. In lobbying, beneficiaries of a piece of legislation who seek to influence legislators typically belong to different industries, each of which has a different stake in the bill's success.

In this paper, we study the ability of such multi-group teams to "get things done." In particular, we are interested in understanding how heterogeneity across groups shapes the expected time to complete projects, or the size of projects that the team can complete. On the behavioral (positive) side, we analyze how heterogeneity across groups affects each group's contribution pattern. Does the group with a higher valuation for completion of the project contribute more? Do agents belonging to this group contribute in earlier or later phases of the project?

We study these questions in a decentralized environment without commitment, where the identity of the contributor and the amount of the contribution are not determined by a central authority or contract, but instead come from random opportunities to add value (e.g. ideas), and individual incentives to cooperate. This is particularly relevant when effort arises organically within horizontal structures. In line with our examples, we treat projects as binary; i.e., payoffs accrue only when the paper is completed, when the product is launched, when the bill passes, etc.

In the model, two groups of internally identical agents – say junior and senior researchers – collaborate on a binary project. We assume that junior and senior agents get a positive payoff (which is allowed to differ across groups) when the project is completed, and zero otherwise. To capture the decentralized nature of contributions, we assume that each period an agent is selected at random to make a contribution to the project. We allow juniors and seniors to have a different probability of having an opportunity to contribute, but assume that it is equal across agents within the same group. To distinguish strategic considerations from technological incentives for gradualism, we assume that the cost of contributing is linear, and assume there are no deadlines, or asymmetric information among agents. To maintain within-group homogeneity, we focus on group-symmetric subgame perfect equilibria (SPE).

We show that the game has a generically unique group-symmetric SPE in pure strategies, and that this equilibrium is Markovian: agents' equilibrium strategies depend only on the remaining effort required to complete the project (i.e., the state). We can then solve the game recursively and show that large projects can be divided into a sequence of smaller sub-projects, each of which is characterized by one group taking ownership of the sub-project while the members of the other group remain inactive or contribute significantly smaller amounts.

Our first main result characterizes the final small sub-project, which is completed without delay by the group whose members have the greatest willingness to finish, which we refer to, for expositional purposes, as the group of juniors. We show that in equilibrium, seniors are only willing to make partial contributions towards completion, moving the project forward in a series of steps, which decrease in size the farther the project is from completion. Remarkably, at some point, seniors are not willing to contribute anymore, while juniors are still willing to finish the project outright. Hence, there is a set of states in which seniors do nothing, free riding entirely on juniors for the completion of the project in one shot. Conceptually, free riding takes two coexisting forms: inaction and gradualism. Gradualism primarily reflects in-group free riding, while inaction stems from out-group free riding. To our knowledge, no existing model simultaneously captures both forms of delay.

Our second main result exploits that larger projects can be divided into a sequence of smaller sub-projects. As with the final sub-project, each stage involves two groups: one that is active, with members willing to complete the task immediately, and another that is sluggish or even inactive. Two notable features emerge. First, the identity of the active group changes across sub-projects, generating contribution cycles, in which juniors and seniors alternate in taking responsibility for moving the project forward (Figure 1). Second, the greater the number of sub-projects required to complete the larger project, the more pronounced free riding becomes in equilibrium. This also results in gradualism: the effort exerted by the member of the active group called to contribute diminishes the further the project is from completion.

These contribution cycles are generally asymmetric: one group tends to be active

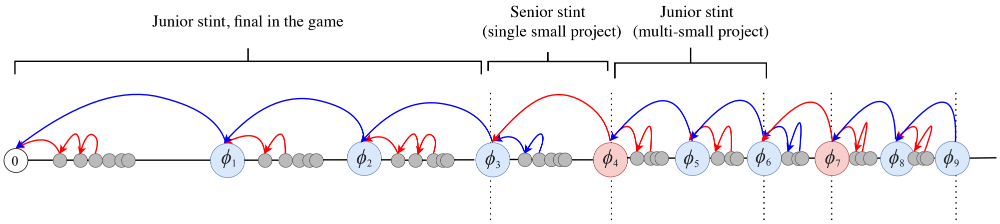

Figure 1: Junior (blue) and senior (red) agents alternate contributing in stints of small projects towards completion of the project.

across multiple consecutive sub-projects, while members of the other group typically contribute in only a single sub-project at a time. Crucially, differences in preference intensity or contribution costs across groups affect only the identity of the group that completes the last sub-project but they do not influence the structure of contribution cycles in earlier phases. These early-phase dynamics are driven purely by free-riding incentives. Specifically, we show that the group undertaking multiple consecutive sub-projects is the one facing stronger in-group free-riding incentives (i.e., the larger group), and weaker out-group free-riding incentives (i.e., the group more likely to receive opportunities to contribute).

Equilibrium outcomes are inefficient not only due to the delays induced by gradualism and *inaction* along the path of play, but also because of more traditional free-riding effects: some welfare-enhancing projects remain incomplete. We show that these inefficiencies grow with group size and we examine how heterogeneity across groups shapes the different forms of free riding.

This leads us to our final contribution, which speaks to an influential observation by Mancur Olson in his landmark work *The Logic of Collective Action* (1965, p. 35):

> This suboptimality or inefficiency [in provision of a collective good] will be somewhat less serious in groups composed of members of greatly different size or interest in the public good. In such unequal groups, on the other hand, there is a tendency toward an arbitrary sharing of the burden of providing the collective good. The largest member, ... who would on his own provide the largest amount of the good, bears a disproportionate share of the burden of providing the collective good.

We find that inefficiencies in public good provision decline as heterogeneity across teams increases, in line with Olson's hypothesis. Specifically, we show that a team composed of two equally sized groups can complete larger projects than a fully homogeneous team, even when the differences in completion preferences between the groups are arbitrarily small. Moreover, for sufficiently large projects, the two-group team always completes the project strictly faster in expectation. However, the burden of contributions is not distributed arbitrarily; instead, it follows a well-defined cyclical pattern. Importantly, this pattern is not driven by differences in how much members value the completion of the public good, as Olson suggests, but rather by differences in individuals' *ability* to contribute.

2 Related Literature

Our paper is related to the literature on dynamic contributions to public goods, particularly how dynamic considerations affect free-riding incentives. Two early papers are Fershtman and Nitzan (1991) and Admati and Perry (1991). Fershtman and Nitzan (1991) consider a problem in which the public good delivers continuous benefits, and study the efficiency of public good provision when all agents can contribute at the same time and contributions are made in continuous time. Admati and Perry (1991), instead, consider a binary project, in which benefits are attained only when the project is completed, only one player can contribute at a time, and contributions can be made in discrete time. Our paper builds in the tradition of the latter.[1]

The distinction between continuous and binary public goods is fundamental for dynamic free-riding incentives. In continuous public goods, contributions are strategic substitutes over time: present contributions lead to a reduction of effort in the future (Kessing (2007)). In binary public goods, instead, contributions are strategic complements over time: a higher contribution by one agent induces larger contributions in the future. This complementarity is a central feature of our model, as well as of Admati and Perry (1991), Compte and Jehiel (2003, 2004), Kessing (2007), Giorgiadis (2015), and Bowen et al. (2019).

Our paper is most closely related to Admati and Perry (1991) and Compte and

[1]Battaglini et al. (2014) and Harstad (2016) also consider continuous public goods, while Marx and Matthews (2000) consider both cases.

Jehiel (2003). Admati and Perry (1991) consider a model in which two identical agents take turns contributing to the public good. When the cost of contributing is strictly convex, the game has an essentially unique SPE path, in which agents make gradual contributions to the project. If the contribution cost is linear, the inefficient equilibrium with gradual contributions still exists, although there is also a no-delay, "one-step completion" equilibrium. Compte and Jehiel (2003) consider a variant of Admati and Perry's model in which agents differ in their valuation of the good and focus on the case of linear costs. In contrast to Admati and Perry's model, in equilibrium, each player makes only one large contribution. Moreover, whenever the project is socially desirable, it is completed as players grow arbitrarily patient. Thus, the gradual equilibrium of Admati and Perry appears not to be robust to asymmetries in agents' valuations.

Our paper considers an environment similar to that in Compte and Jehiel (2003), but with multiple agents of each type, and no pre-specified order of play. We show that cycles emerge endogenously in equilibrium if the project is large enough. In this equilibrium, agents make partial contributions to the public good, recovering the gradualism in Admati and Perry (1991), and, in the special case in which there is exactly one agent of each type, our model produces the result of Compte and Jehiel (2003) in which the project is completed in two steps. We conclude that gradualism and alternation can arise purely as strategic behavior when multiple groups, each with internally homogeneous but mutually distinct free-riding incentives, interact. The emergence of endogenous contribution cycles is understudied, and in particular, the emergence of repeated cycles of endogenous length is, to the best of our knowledge, new to the literature.

Gradualism in public good provision emerges under various conditions in binary public good contribution games. It arises naturally when effort costs are convex (Admati and Perry (1991) and Kessing (2007)); when costs are linear and there are history-dependent outside options (Compte and Jehiel (2004)); and even under private information (Marx and Matthews (2000)).

When agents can contribute simultaneously and effort unfolds in continuous time, gradualism also arises in equilibrium.[2] In Giorgiadis (2015), as in our paper, contributions are strategic complements. In contrast to our paper, the project progresses

[2]In some of the equilibria in Battaglini et al. (2014), gradualism is central to achieving efficiency.

stochastically in continuous time. Focusing on the Markov perfect equilibrium, he shows that effort levels increase as the project approaches completion. In contrast to our results, however, larger teams in his model exert greater effort early in the project, due to an "encouragement effect" generated by faster expected completion.[3] In Bonatti and Hörner (2011), by contrast, contributions act as strategic substitutes. They study a dynamic contribution game in continuous time where the project's viability is uncertain. In their equilibrium, players grow increasingly pessimistic over time, which leads to declining effort. None of these papers obtain inaction in equilibrium with endogenous cycles, as in our model.

3 The Model

We consider a model in which n agents collaborate to produce a public good, which requires q units of investment. There is an infinite horizon, $t = 1, 2, \ldots$. In each period t, an agent $i(t)$ selected at random has the opportunity to invest toward the public good. Contributing $e \geq 0$ units has a cost $C(e) = ce$, for $c > 0$.[4] We let $e^i(t)$ denote the contribution made by agent i in period t ($e^i(t) = 0$ for all $i \neq i(t)$) and $e(t)$ denote the contribution made in period t by the agent selected in that period. We denote the amount of effort outstanding for completion of the good at the end of period t by $s_t = \max\{s_{t-1} - e(t), 0\}$, with $s_0 = q$; i.e., at the beginning of the game, the group is q units of effort away from completion.

There are two groups of agents, $m = 1, 2$, and $n^m > 1$ agents of type $m = 1, 2$ (we cover single-member groups as special cases). The probability that any particular individual of group m is selected is $\tilde{\pi}^m/n^m$, where $\tilde{\pi}^1 + \tilde{\pi}^2 = 1$; i.e., group m has probability $\tilde{\pi}^m$ of being selected, and conditional on group m being recognized, each member of that group has the same chance of being called to contribute. The payoff of a group-m agent i is:

$$U^i = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \left[u^m(s_t) - ce^i(t) \right],$$

[3] Bowen et al. (2019) examine a related setting, but in the context of a collective choice problem with heterogeneous agents.

[4] With a convex cost, gradualism may be optimal, inducing cycles for technological reasons and not for strategic reasons. We restrict attention to linear costs to isolate strategic incentives.

where $u^m(s) = 0$ for $s > 0$ and $u^m(0) = \theta^m > 0$; i.e., agents get no benefits until the project is finished, and upon completion, members of group m enjoy a payoff θ^m.

Agents are heterogeneous along several dimensions. We now introduce a parameter that will prove useful in the analysis below: the largest project that a type m agent would be willing to complete outright, even when she knows that all other agents would also complete the project whenever given the opportunity. That is, it is the project size at which free-riding considerations are irrelevant for the behavior of a type m agent. This non-equilibrium quantity is given by:

$$w^m(1) \equiv \left(\frac{n^m}{n^m - \delta\tilde{\pi}^m}\right)\frac{\theta^m}{c} \tag{3.1}$$

for $m = 1, 2$, and we make the (generic) expositional assumption

Assumption 3.1.

$$w^2(1) < w^1(1)$$

In Lemma 4.1, we show that a direct implication of this inequality is that group 1 has stronger incentives to complete the project when it is close to completion. When the individual recognition probability is the same for all players, i.e., $\tilde{\pi}^1/n^1 = \tilde{\pi}^2/n^2$, the condition $w^2(1) < w^1(1)$ simplifies to group 1 assigning a higher value to the project than group 2; that is, $\theta^1 > \theta^2$. Conversely, if the intrinsic value of project completion is identical for both groups, i.e., $\theta^1 = \theta^2$, the same condition implies that members of group 2 have a lower probability of recognition; that is, $\tilde{\pi}^1/n^1 > \tilde{\pi}^2/n^2$. For ease of exposition, we refer to group 1 as *juniors* and group 2 as *seniors*. That is, juniors may be more motivated to complete the project, perhaps due to career concerns, or more likely to contribute, possibly because they are involved in fewer side projects than seniors.

A time-t *history* $h(t)$ is the tuple $(\{i(\tau)\}_{\tau=1}^{t-1}, \{e(\tau)\}_{\tau=1}^{t-1})$, which encodes previous game play, and the remaining units needed for completion, $s^{h(t)} = q - \sum_{\tau=1}^{t-1} e(\tau)$. Let H^t be the set of all time-t histories, and $H := \cup_{t\in\mathbb{N}} H^t$ be the set of all histories in the game. A pure strategy for a group-m agent i is a function $\sigma^m(\cdot) : H \to \mathbb{R}_+$, where $\sigma^m(h)$ denotes the contribution of an agent of group-m from history h when she has the opportunity to contribute. An *equilibrium* is a (group) symmetric subgame perfect equilibrium in pure strategies.

Consider any history h and let h_e be the history induced by group-m agent contributing $e \in [0, s^h]$ at h. We denote the value of a player of type m at history h by $V^m(h)$, and the payoff for a player of type m at history h when this player is called to contribute and contributes e by $W^m(e, h)$. Since the current period (flow) utility is given by $u^m(s^h - e) - ce$, we have that

$$W^m(e, h) = \delta V^m(h_e) + (1 - \delta) \left[u^m(s^h - e) - ce \right]. \tag{3.2}$$

We formally define our notion of equilibrium discussed above, as follows.

Definition 3.2. *A (group) symmetric subgame perfect equilibrium in pure strategies is a pair* $(\sigma^1(\cdot), \sigma^2(\cdot))$ *such that* $\sigma^m(h) = \arg\max_{e \geq 0} W^m(e, h)$ *for all* $m = 1, 2$ *and all histories* h *with* $s^h \leq q$, *and* $V^m(h)$ *satisfies* (i) $V^m(h) = \theta^m$ *for any* $h \in H$ *for which* $s^h = 0$, *and* (ii) *for any* h *with* $s^h > 0$,

$$V^m(h) = \sum_{j=1}^{2} \tilde{\pi}^j \left[\delta V^m(h_{\sigma^j(h)}) + (1 - \delta) u^m(s^h - \sigma^j(h)) \right] - \frac{\tilde{\pi}^m}{n^m} c(1 - \delta) \sigma^m(h). \tag{3.3}$$

Our definition of equilibrium uses the continuation values (3.3) induced in equilibrium. This equation highlights the free riding at $h(t)$: the probability associated with the continuation value $V^m(h_{\sigma^j(h)})$, differs from the probability associated with the cost if called to contribute $c(1 - \delta)\sigma^m(h)$. Strictly speaking, this object is not formally needed but it will be useful later on, when we show that there is a unique (group) symmetric subgame perfect equilibrium and this equilibrium is history independent.

4 Results

In this section, we present our results. We begin by analyzing "small" projects in Section 4.1. We define a project as small if it starts at a history h such that there exists an equilibrium in which the project is completed immediately with positive probability. We denote the size of the largest small project by ϕ_1. After characterizing the equilibria of small projects, we turn to our main analysis in Section 4.2. We show that any project of size $q > \phi_1$ can be viewed as a sequence of sub-projects.

Each sub-project has a structure similar to that of the final small project, where the associated payoffs are given by the continuation values upon completion of each sub-project. Thus, the equilibrium can be characterized recursively as a sequence of *small projects*. Using these tools, we characterize equilibrium outcomes in large projects, in particular contribution cycles and efficiency.

4.1 Small Projects

We first establish that in equilibrium, there exists a unique threshold $w(1) > 0$ such that the project is completed without delay if and only if the remaining units needed for completion, $s^{h(t)} = q - \sum_{\tau=1}^{t-1} e(\tau)$, are no larger than $w(1)$. We also show that equilibrium strategies from any history h with the same $s^h = s$ are *history-independent*, and uniquely determined by s. Thus, sufficiently close to completion (i.e., in any small project), there is a unique equilibrium, which is Markov when the state is defined as the remaining units needed for completion. Using this fact we refer to the remaining units needed for completion, $s^{h(t)} = q - \sum_{\tau=1}^{t-1} e(\tau)$, as the corresponding state for $h(t)$.

Lemma 4.1. *In equilibrium, $\sigma^m(h) = s^h$ for all $m \in \{1,2\}$ if and only if $s^h \leq w(1) = w^2(1)$.*

The logic for the result is straightforward. First, given any history h, when s^h is sufficiently small, it is a dominant strategy for any agent to finish the project outright.[5] Then, there exists a *maximal* state, which we call $w(1)$, such that any agent of any group is willing to complete the project outright from any history h with $s^h \leq w(1)$, given that all other agents do the same. Therefore, outright completion by any agent occurs if and only if $s^h \leq w(1)$. Not surprisingly, $w(1) \equiv \min\{w^1(1), w^2(1)\}$ and, by assumption 3.1 we have $w(1) = w^2(1)$.

As shown above, seniors do not immediately finish the project when they have an opportunity to play from any history with $s^h > w(1)$. Yet, because contributions are strategic substitutes, there exists $s' > w(1)$ such that in any history h' with $s^{h'} \leq s'$, any junior still completes the project immediately whenever she is recognized. Recall

[5] This is intuitive: if the agent does not finish outright, she can at best expect another agent to finish the project tomorrow for sure. Thus, the payoff for waiting is at most $\delta\theta^m$. On the other hand, finishing outright gives a payoff $\theta^m - c(1-\delta)s^h$ which is higher than $\delta\theta^m$ whenever $cs^h < \min\{\theta^1, \theta^2\}$.

that ϕ_1 denotes the largest remaining contribution needed for completion such that in any history h with $s^h \leq \phi_1$, juniors finish the project outright in equilibrium.[6] We show that there exists some threshold $\overline{s}_1 \leq \phi_1$ such that for all histories h with $s^h < \overline{s}_1$, seniors gradually move the project towards completion when they have an opportunity to play. This gradualism emerges because equilibrium incentives must balance the trade-off between the cost of delayed completion and the incentives to free ride. In equilibrium, seniors follow a *cutpoint strategy*: the interval of remaining contributions before reaching $w^2(1)$, namely $\overline{s}_1 - w^2(1)$, is partitioned into small segments such that, whenever a senior is called to contribute, she contributes only enough to reach the next segment.[7] Remarkably, though, this process *must come to a halt at a point $\overline{s}_1$ strictly below ϕ_1*. Thus, there is a region $[\overline{s}_1, \phi_1]$ within which seniors do not contribute at all, while juniors finish the project outright. This is an extreme form of free riding from seniors, who free ride entirely on juniors finishing the project outright at any history with $s^h \in [\overline{s}_1, \phi_1]$.

Proposition 4.2. *In equilibrium,*

1. *Juniors finish the project outright at all histories with $s^h \leq \phi_1$; i.e., $\sigma^1(h) = s^h$ for all $s^h \leq \phi_1$, where*

$$\phi_1 = \left(\frac{1}{1 - \delta + \delta \tilde{\pi}^1 \left(\frac{n^1 - 1}{n^1} \right)} \right) \frac{\theta^1}{c}. \tag{4.1}$$

2. *Seniors follow a cutpoint strategy in state h with $s^h \in [0, \overline{s}_1] \subset [0, \phi_1]$ with cutpoints given by*

$$s_1(k) = \frac{1 - \left(\delta \tilde{\pi}^2 \left(\frac{n^2 - 1}{n^2 - \delta \tilde{\pi}^2} \right) \right)^k}{1 - \delta \tilde{\pi}^2} \frac{\theta^2}{c} < \overline{s}_1 \equiv \left(\frac{1}{1 - \delta \tilde{\pi}^2} \right) \frac{\theta^2}{c} \quad \forall k \geq 1, \tag{4.2}$$

[6] The equilibrium object ϕ_1 exceeds the parameter $w^1(1)$. Recall that $w^1(1)$ is derived under the hypothetical scenario in which *all* agents complete the project outright. However, when $s > w^2(1)$, seniors free ride, which increases the effective cost of delay faced by juniors relative to that hypothetical benchmark. As a result, juniors are willing in equilibrium to contribute more to ensure project completion than they would under the (non-equilibrium) benchmark in which no agent free rides.

[7] We say that σ^m is a cutpoint strategy in $[a, b] \subseteq S$ if there is a strictly increasing sequence $\{s(k)\}_{k=0}^K$, with K possibly infinite, such that (i) $s(0) \equiv a$, and $\lim_{k \to K} s(k) = b$, and (ii) $\sigma^m(s) = s - s(k-1)$, $\forall s \in (s(k-1), s(k)]$ and $\forall k : 1 \leq k \leq K$.

such that for any h with $s^h \in (s(k-1), s(k)]$,

$$\sigma^2(h) = s^h - s(k-1)$$

and at all h with $s^h > \overline{s}_1$, do not contribute ($\sigma^2(s^h) = 0$).

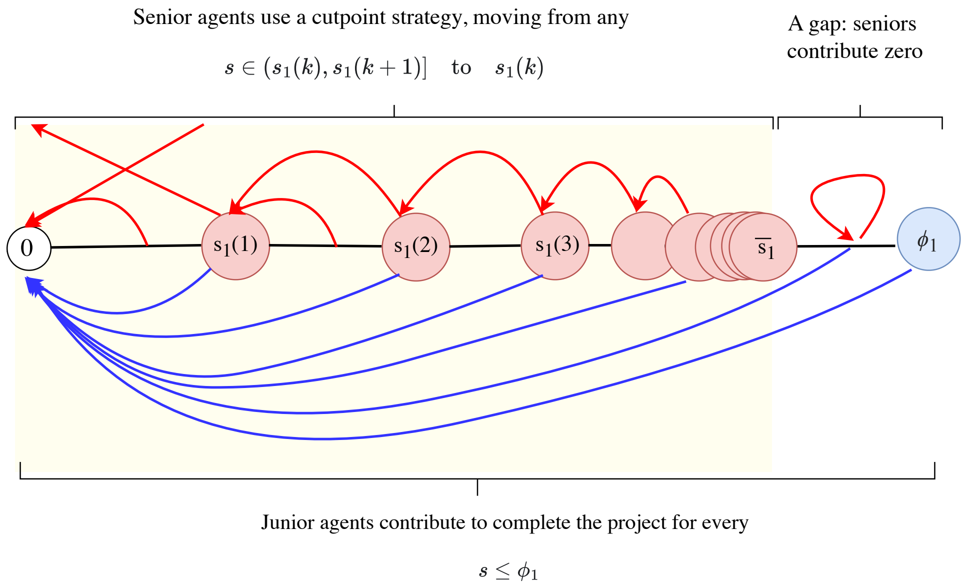

Figure 2: Equilibrium in Small Projects, in the generic case $w^1(1) > w^2(1)$.

Because all players have a dominant strategy for every $s < w(1)$, there is essentially no scope for punishment. Moreover, when attention is restricted to group-symmetric strategies, there is correspondingly little scope for discriminating between deviators and non-deviators. Since delay costs are discrete, optimal contributions must be made in fixed increments. Hence, each member has a maximal contribution level they are willing to provide. Backward induction then implies that the equilibrium is unique. Moreover, the unique equilibrium for small projects, characterized above in Proposition 4.2, is Markov in the state. We highlight this important fact in the following corollary.

Corollary 4.3. *From any history h with $s^h \leq \phi_1$, there exists a unique equilibrium, which is Markov in the state.*

Since the equilibrium is Markovian, we use the expressions "from state s" or "at s" to mean "from any history in which the remaining effort required for completion is given by s." We also slightly abuse notation to write $\sigma^m(s)$ as the strategy played by, and $V^m(s)$ as the value to, any group-m agent from any h with $s^h = s$.

To understand the determination of the cutpoints in part (2) of Proposition 4.2, we combine the equilibrium (continuation) values and the individual incentives of the player who is recognized to contribute in a given period. For the latter, note that for all k in the sequence, and all $s \in (s_1(k-1), s_1(k)]$, the value (3.3) for a group-m agent is

$$V^m(s) = \tilde{\pi}^1\theta^m + \tilde{\pi}^2\delta V^m(s_1(k-1)) - \frac{\tilde{\pi}^m}{n^m}c(1-\delta)\sigma^m(s) \tag{3.3b}$$

In equilibrium, juniors finish the project, so with probability $\tilde{\pi}^1$ the project is completed immediately leading to a payoff of θ^m. Seniors, on the other hand, play a cutoff strategy, so with probability $\tilde{\pi}^2$ the project is only moved to the next cutoff leading to a payoff of $\delta V^m(s_1(k-1))$. With probability $\tilde{\pi}^m/n^m$, a player type m is recognized to contribute leading to the cost $c(1-\delta)\sigma^m(s)$.

Next, for the individual incentives of the player called to contribute, consider an arbitrary cutpoint $s_1(k)$ in the sequence. Note that at $s = s_1(k) + \epsilon$, seniors must prefer contributing $\epsilon > 0$ to overshooting to any other state $s' < s_1(k)$. Similarly, at $s = s_1(k) - \epsilon$, seniors must prefer to contribute in order to reach the cutoff $s_1(k-1)$ instead of staying in state $s_1(k) - \epsilon$.[8] Taking the limit as $\epsilon \to 0$ we get

$$\delta V^2(s_1(k)) = \delta V^2(s_1(k-1)) - c(1-\delta)(s_1(k) - s_1(k-1)). \tag{4.3}$$

That is, the senior who is called to contribute, is indifferent between *doing her part* and doing nothing and delaying the completion of the project: the extent to which seniors can free ride in small projects is given by their tolerance to creating delay.

Combining (4.3) with (3.3b) for $m = 2$ gives

$$s_1(k) = \frac{n^2}{n^2 - \delta\tilde{\pi}^2}\frac{\theta^2}{c} + \frac{\delta\tilde{\pi}^2}{n^2 - \delta\tilde{\pi}^2}(n^2 - 1)s_1(k-1). \tag{4.4}$$

[8]Technically, for all $s \in (s_1(k-1), s_1(k)]$ the value of the senior contributor $W^2(s) = \delta V^2(s - \epsilon) + (1-\delta)c\epsilon$ is decreasing in ϵ for any $\epsilon < s - s_1(k-1)$ and it jumps at $\epsilon = s - s_1(k-1)$, where it starts to decrease again.

Solving this difference equation allows us to characterize the equilibrium values $V^m(s)$ for both types, for all $s \leq \lim_{k\to K} s_1(k)$. On the other hand, an immediate implication of the contribution "gap" in Proposition 4.2 is that the values at the "beginning" of the small project are independent of the details of seniors' cutpoint strategy, and given by

$$\delta V^1(\phi_1) = \alpha^1\theta^1 \quad \text{and} \quad \delta V^2(\phi_1) = \beta^1\theta^2, \tag{4.5}$$

where

$$\alpha^m \equiv \frac{\delta\tilde{\pi}^m \frac{n^m-1}{n^m}}{1-\delta+\delta\tilde{\pi}^m \frac{n^m-1}{n^m}} \quad \text{and} \quad \beta^m \equiv \frac{\delta\tilde{\pi}^m}{1-\delta+\delta\tilde{\pi}^m}$$

are the *effective discount rates* defined so that, when group m is active, α^m reflects both the cost of delay and the expected cost of contributions, whereas β^m reflects only the cost of delay. Because delay is costly when $\delta < 1$ and $\tilde{\pi}^m < 1$ we have $\alpha^m < \beta^m < \delta$.

To understand the inaction gap ($\overline{s}_1 < \phi_1$), let $\hat{k} \in \{1, 2, \dots, \infty\}$ be the number of senior cut-points below ϕ_1. As $\hat{k}$ increases, the incentives for juniors to complete the project are compounded, since seniors take progressively smaller steps. A junior thus becomes *more* willing to contribute to avoid additional expected periods of delay were she not to. The intuition for this result lies in the asymmetry in contribution strategies between the two groups in equilibrium: while seniors remove only a small amount of delay by contributing to their next cutpoint $s_1(\hat{k})$, a contribution from juniors completes the project outright. Because the marginal value of a contribution relative to shirking is far higher for juniors than seniors, their willingness to increase their contribution as $\hat{k}$ increases is also larger than the contribution seniors are willing to make.

Delay and in-group and out-group effects. At the core of these results is the tradeoff between delay and free riding on both peers – *in-group* effect– and outsiders –*out-group* effect–. Using Proposition 4.2, we can compute the expected time for completion of any project $q < \overline{s}_1$ as,

$$\mathcal{E}(q) = \begin{cases} (1-(\tilde{\pi}^2)^{\psi(q)})/\tilde{\pi}^1 & \text{if } q < \overline{s}_1 \\ 1/\tilde{\pi}^1 & \text{if } q \in [\overline{s}_1, \phi_1] \end{cases} \tag{4.6}$$

where

$$\psi(q) \equiv \left\lceil \log\left(1 - \frac{cq}{\theta^2}(1-\delta\tilde{\pi}^2)\right) \Big/ \log\left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)\right\rceil$$

is the total number of steps required for completion of a project of size $q < \overline{s}_1$ (see 4.2). It follows that

Proposition 4.4. *For any $q < \phi_1$, expected delay generically decreases with $\tilde{\pi}^1$.*

Note that $\psi(q)$, and therefore $\mathcal{E}(q)$, are weakly increasing in the total cost of the project cq, and weakly decreasing in the value that senior agents put on completion of the project, θ^2. Moreover, the expected delay for $q < \overline{s}_1$ is also weakly increasing in the number of senior agents in the group. Interestingly, expression (4.6) shows that there exists an upper bound on expected delay for any $q \leq \phi_1$, given by $1/\tilde{\pi}^1$.

The intuition for these results is driven by changes in the incentives to free ride. Consider first the states in which seniors do not contribute, $s \in (\overline{s}_1, \phi_1]$, and recall that the relevant equilibrium objects depend on parameters such as $\tilde{\pi}^1$ and n^1. An increase in either $\tilde{\pi}^1$ or n^1 raises the probability that the current junior will be able to free ride on a junior who is called to contribute tomorrow. This reduces the cost of shirking for the junior who is responsible for contributing today, thereby weakening their incentives to contribute. To preserve incentive compatibility in equilibrium, the contribution required from the current junior must therefore decrease, which results in a lower value of ϕ_1.

Consider now the states in which seniors do contribute: i.e. $s < \overline{s}_1$. Juniors' behavior is not affected by any change in parameters as they strictly prefer to finish the project outright. Increasing $\tilde{\pi}^1$ increases the chances of free riding on a junior called to contribute tomorrow so, again, to maintain equilibrium incentives due to reduced expected delay and lower costs of shirking, the size of the contribution for the senior called to contribute today decreases. This leads to a smaller distance between the cutoffs $s_1(k) - s_1(k-1)$ as well as a reduction in ϕ_1. This is the pure out-group effect. On the other hand, the pure *in-group* effect, driven by the number of seniors n^2, is more involved. Intuitively, increments in n^2 increase the incentives to free ride as free riding on peers is more attractive. This is easy to see at $w^2(1)$ which is decreasing in n^2. Note that $\underline{s}^1$ is not affected by changes in n^2. This implies that, at some point, the distance between cutoffs increases. In fact, the distance $s_1(k) - s_1(k-1)$ increases with n^2 if and only if $k < 2$. This is because the increase in expected delay caused by additional free riding compounds with the associated increase in the cost of shirking, an effect that becomes stronger farther from completion.[9]

4.2 Large Projects

We now study the equilibrium of the contribution game for any possible project length q. Before turning to the formal characterization of the equilibrium, we present a verbal overview, referring to Figure 1 for illustration. Large projects of size q can be viewed as a sequence of $\overline{\tau}$ smaller sub-projects, where each sub-project has a size of $\phi_\tau - \phi_{\tau-1}$, with $\phi_{\overline{\tau}} = q$. Each sub-project $\tau \leq \overline{\tau}$ is associated with an *active group*, denoted $m_\tau \in \{1, 2\}$. Members of the active group complete the sub-project outright, contributing enough to reach the next cut point $\phi_{\tau-1}$. In contrast, the behavior of members of the non-active group mirrors what was previously described for the final sub-project, with sluggishness and inaction: they provide partial contributions when they are close to completing the sub-project and remain inactive when they are too far from its completion.

The identity of the active group and the number of sub-projects required to complete the large project are both determined endogenously. The core problem is to identify which group is active in each sub-project. The change in identity of the active group will follow from the fact that the active group discounts future outcomes more steeply, i.e., $\alpha^m < \beta^m$, causing the perceived value to decline faster than for the non-active group. This difference in intertemporal valuation is the source of the cyclical behavior we observe. The main result of this section is then:

Theorem 4.5. *There exists an almost everywhere unique equilibrium in pure strategies. The equilibrium is characterized by an increasing sequence $\{\phi_\tau\}_{\tau=0}^{\overline{\tau}}$, with $\phi_0 = 0$ and $\overline{\tau}$ possibly infinite, such that:*

(i) For any $\tau \geq 1$, there is a unique $m_\tau \in \{1, 2\}$ such that

(a) $\sigma^{m_\tau}(s) = s - \phi_{\tau-1}$ for all $s \in (\phi_{\tau-1}, \phi_\tau]$, and

[9]Other models show similar non-monotonicities regarding the state of the project. See for example, Proposition 1 in Giorgiadis (2015), describing a form of *encouragement effect*.

(b) σ^{-m_τ} *is a cutpoint strategy in* $[\phi_{\tau-1}, \overline{s}_\tau]$ *for* $\overline{s}_\tau < \phi_\tau$, *and* $\sigma^{-m_\tau} = 0$ *in* $(\overline{s}_\tau, \phi_\tau]$.

(ii) *The sequence* $\{\phi_\tau\}$ *converges to a limit* $\tilde{\phi} > 0$, *and for* $\tau > 1$,

$$\phi_\tau - \phi_{\tau-1} = \left(\frac{1}{1 - \delta + \delta \tilde{\pi}^{m_\tau} \frac{n^{m_\tau} - 1}{n^{m_\tau}}} \right) \frac{\delta V^{m_\tau}(\phi_{\tau-1})}{c}. \tag{4.7}$$

(iii) *If* $q < \tilde{\phi}$, *the project is finished in finite time, and otherwise, the project is never started; i.e.,* $\sigma^m(s) = 0 \ \forall s \in [\tilde{\phi}, q]$ *and* $m \in \{1, 2\}$.

The basic intuition for this result relies on two key facts. First, note that at any $s > \phi_1$, agents of both groups are not willing to contribute more than the needed amount to reach ϕ_1. It follows that no agent moves the project beyond ϕ_1 at any $s > \phi_1$ (see Lemma A.1 in the Appendix). This implies that for the project to be completed from $s > \phi_1$, the project reaches ϕ_1 at some point and has to pass through ϕ_1 on the equilibrium path. It follows then that the game can be truncated at ϕ_1 given by (4.1) by considering ϕ_1 as the terminal state, with payoffs $\delta V^m(\phi_1)$ when ϕ_1 is attained.

Second, recall that the equilibrium characterization in Proposition 4.2 relied solely on the labeling assumption that $w^2(1) < w^1(1)$. In a similar fashion, when considering the truncated game at ϕ_1, we define

$$w^m(2) \equiv \frac{n^m}{n^m - \delta \tilde{\pi}^m} \frac{\delta V^m(\phi_1)}{c} = w^m(1) \frac{\delta V^m(\phi_1)}{\theta^m} \tag{4.8}$$

Using Proposition 4.2 we obtain that the active group in the second to last small sub-project is given by $m_2 = \{m = 1, 2 : w^{-m}(2) < w^m(2)\}$. Equilibrium behavior in this small project is described by replacing juniors (group 1) with group 2, $w^m(1)$ with $w^m(2)$, and θ^m with $\delta V^m(\phi_1)$. In fact, for any $\tau < \overline{\tau}$, we have

$$w^{m_\tau}(\tau + 1) = \alpha^{m_\tau} \ w^{m_\tau}(\tau) \quad \text{and} \quad w^{-m_\tau}(\tau + 1) = \beta^{m_\tau} \ w^{-m_\tau}(\tau) \tag{4.9}$$

Since $\alpha^{m_\tau} < \beta^{m_\tau}$, the equilibrium payoff of non-active contributors only decreases with τ due to impatience and delay while the equilibrium payoff of active contributors decreases with τ due to impatience, delay, *and* due to the cost of moving the project forward. This will be the source of the changes in the identity of the active group.

Note that generically, $w^1(\tau) \neq w^2(\tau)$ for any $\tau \geq 1$. This allows us to characterize the equilibrium of the full game by sequentially truncating it at each state ϕ_τ, which defines the upper bound of sub-project τ for a given $\phi_{\tau-1}$. By continuing this process and exploiting the recursive structure, parts *(i)* and *(ii)* of the theorem follow immediately from our analysis in Section 4.1.

4.2.1 Endogenous Contribution Cycles

In this section, we explain the emergence of contribution cycles. Large projects can be divided into a sequence of stints $k = 1, 2, \ldots$. A stint is a connected set of sub-projects with the same active group. Each sub-project within a stint is referred to as a *step*, and we denote by $j(k)$ the number of steps in stint k so, by definition $\overline{\tau} = \sum_k j(k)$. We show that contribution cycles follow a regular pattern, with the possible exception of the first stint $(\overline{\tau})$. One group is active in stints consisting of at most a single step, $j(\cdot) = 1$, whereas the other group tends to be active in stints that last for multiple steps, $j(\cdot) > 1$. Moreover, when this group is active, the number of steps in its stints is typically constant, though occasionally an additional step may occur.

From Theorem 4.5, there is generically a unique contributor $m_\tau \in \{1, 2\}$ at each cutpoint ϕ_τ. From (4.5), and using $\delta V^m(\phi_1)$ as the terminal payoffs in the truncated game, for all $\tau > 1$ we have:

$$\frac{V^{m_\tau}(\phi_\tau)}{V^{-m_\tau}(\phi_\tau)} = \frac{\alpha^m}{\beta^m} \frac{V^{m_\tau}(\phi_{\tau-1})}{V^{-m_\tau}(\phi_{\tau-1})} \tag{4.10}$$

and from (4.9) we get

$$\frac{w^{m_\tau}(\tau+1)}{w^{-m_\tau}(\tau+1)} = \frac{\alpha^{m_\tau}}{\beta^{m_\tau}} \frac{w^{m_\tau}(\tau)}{w^{-m_\tau}(\tau)} < \frac{w^{m_\tau}(\tau)}{w^{-m_\tau}(\tau)}$$

Note that for any $\tau \geq 1$, an agent of group m is the active contributor for step τ if and only if $w^m(\tau)/w^{-m}(\tau) > 1$. Although $w^{m_\tau}(\tau)/w^{-m_\tau}(\tau) > 1$, the more steps group m_τ remains active the lower $w^{m_\tau}(\tau+1)/w^{-m_\tau}(\tau+1)$ becomes. It is easy to see that the identity of the active group in stint k starting with sub-project τ (state

ϕ_τ) then switches at $\tau + j(k)$ such that

$$j(k) \equiv \inf\left\{z \in Z^+ : \frac{w^{m_\tau}(\tau+z)}{w^{-m_\tau}(\tau+z)} < 1\right\} = \left\lfloor 1 + \frac{\log\left[w^{m_\tau}(\tau)/w^{-m_\tau}(\tau)\right]}{\log\left[\beta^{m_\tau}/\alpha^{m_\tau}\right]}\right\rfloor. \quad (4.11)$$

For example, considering our assumption that $w^1(1) > w^2(1)$, a direct application of (4.11) gives the first result about cycles,

Remark 4.6. *The equilibrium has no contribution cycles if and only if*

$$q < \frac{1-(\alpha^1)^{j(1)}}{1-\delta}\frac{\theta^1}{c} = \frac{1-(\alpha^1)^{j(1)}}{1-\alpha^1}\phi_1 = \phi_{j(1)}.$$

Provided $q \geq \phi_{j(1)}$, the final stint of $\lfloor 1 + (\log\left[w^1(1)/w^2(1)\right] / \log\left[\beta^1/\alpha^1\right])\rfloor$ steps in which juniors are active contributors is preceded by stint 2 of $j(2)$ steps in which seniors are active contributors. How does this process evolve for q large? In other words, how does $j(k)$ change with k? Again, this is just an application of (4.9) and (4.11) multiple times. For example:

$$j(2) = \left\lfloor 1 + j(1)\frac{\log\left[\beta^1/\alpha^1\right]}{\log\left[\beta^2/\alpha^2\right]} + \frac{\log\left[w^2(1)/w^1(1)\right]}{\log\left[\beta^2/\alpha^2\right]}\right\rfloor.$$

This alternation between juniors and seniors in the role of active contributors continues until the associated sequence of contributions $\phi_\tau - \phi_{\tau-1}$ in each step τ converges to zero, or the cumulative contributions reach q (whichever happens earlier):

Theorem 4.7 (Contribution Cycles)**.** *For any integer $k \geq 1$, let $\iota(k) = 1$ if k is odd, $\iota(k) = 2$ if k is even, and define recursively the total number of steps up to and including stint k as $J(k) \equiv J(k-1) + j(k)$, letting $J(0) = 0$. Then for all $k \geq 1$:*

$$m_\tau = \begin{cases} 1 & \text{if } J(2k-2) < \tau \leq J(2k-1) \\ 2 & \text{if } J(2k-1) < \tau \leq J(2k), \end{cases}$$

where the number of steps in stint k, $j(k)$, is given recursively by:

$$\frac{\omega^{\iota(k)}(\phi_{J(k)})}{\omega^{-\iota(k)}(\phi_{J(k)})} = [\Delta^{\iota(k)}]^{j(k)}\frac{\omega^{\iota(k)}(\phi_{J(k-1)})}{\omega^{-\iota(k)}(\phi_{J(k-1)})} \qquad \text{and} \qquad \Delta^{\iota(k)} < \frac{\omega^{\iota(k)}(\phi_{J(k)})}{\omega^{-\iota(k)}(\phi_{J(k)})} < 1, \quad (4.12)$$

where $\Delta^m \equiv \alpha^m/\beta^m$

Theorem 4.7 shows that in the generically unique equilibrium, if there is some τ such that $\phi_\tau < q$, agents belonging to distinct sub-groups within the team endogenously alternate actively contributing to the public good. In general, each contribution stint k can consist of multiple steps. However, in the special case in which $\Delta^1 = \Delta^2 = \Delta$ (i.e. same in and out-group effects) with the possible exception of the first stint, juniors and seniors take turns completing a *single* step. When instead $\Delta^1 \neq \Delta^2$, to restore equilibrium incentives, the length of contribution stints by each group must be asymmetric. In our next result, we characterize the asymmetry in contribution cycles.

Proposition 4.8. *If* $\Delta^{m'} < \Delta^m$, *contribution cycles alternate between single-step stints by type* m' *agents and stints of either* x^m *or* $x^m + 1$ *steps by group-*m *agents, where*

$$x^m \equiv \left\lfloor \frac{\log[\Delta^{m'}]}{\log[\Delta^m]} \right\rfloor$$

To our knowledge, our paper is the first to demonstrate the emergence of endogenous contribution cycles in equilibrium and provide a complete characterization of the pattern of contributions.

Recall that discounting and random recognition generate incentives for players to contribute in "chunks" when they are called upon to act. Along the equilibrium path, the group facing stronger incentives becomes active and makes contributions, allowing the other group of players to stay inactive. In turn, the inactivity of these players sustains the incentives of the active group to continue contributing during their stint. The equilibrium size of each contribution balances these opposing incentives.

However, inactivity also creates a tension. While remaining on the sidelines, the incentives to contribute of the inactive group increase as the active group moves closer to completing the project. Moreover, these incentives grow faster than those of the active players, since inactive players are not currently responsible for contributing. In equilibrium, the size of contributions adjusts so as to align these incentives while keeping the number of contributions within each stint relatively stable.

In-group and Out-group Effects. Two points are worth emphasizing. First, note that the length of each stint $k > 1$ *does not depend on relative preference intensity.*[10] This is because in equilibrium, the last contribution stint by juniors eliminates all differences in willingness to pay across members of both groups due to differences in preference intensity. As a result, in earlier stages of the game, the determination of the identity of the active contributor solely depends on free-riding incentives, as captured by n^m and $\tilde{\pi}^m$. Second, note that if both groups have equal size, then $\Delta^1 > \Delta^2$ if and only if juniors are more likely to be selected to contribute ($\tilde{\pi}^1 > 1/2$). Similarly, if both groups are equally likely to contribute, then $\Delta^1 > \Delta^2$ if and only if juniors outnumber seniors.

The first case of equal size balances out in-group free-riding incentives. Proposition 4.8 says that the group of agents who are active contributors in stints of multiple steps is the group for which out-group free-riding incentives are *smaller*. The second case where $\tilde{\pi}^1 = \tilde{\pi}^2$, balances out-group effects across groups. Proposition 4.8 says that the group of agents who are active contributors in stints of multiple steps is precisely the group for which in-group free-riding incentives are larger. This is because a larger group size means a smaller expected dent in the value of the active contributors in each step, since other agents of the same group can also foot the bill. Thus, it takes longer stints to achieve balance in the value ratio.

Single-Agent Teams. So far, we have assumed that $n^m > 1$ for all $m \in \{1, 2\}$. The basic structure of equilibrium remains essentially unchanged if there is only one agent in any group. However, equilibrium outcomes have some noteworthy features, resulting from the elimination of in-group free riding.

Suppose $n^1 = n^2 = 1$ so in-group free riding is completely shut down. From our analysis in Section 4.1, the equilibrium for small projects in this case is characterized by cutpoints

$$s_1(1) = \frac{\theta^2/c}{1 - \delta\tilde{\pi}^2} = \overline{s}_1 \quad \text{and} \quad \phi_1 = \frac{\theta^1/c}{1 - \delta}$$

with both players finishing the project outright if $s \leq s_1(1)$ and only juniors finishing the project if $s \in (s_1(1), \phi_1]$. Having a single senior member, $n^2 = 1$, minimizes

[10] This observation, together with the defining pattern of cycles, refines and qualifies Mancur Olson's quote presented in the Introduction.

sluggishness as a form of gradualism in small projects, and having a single junior member, $n^1 = 1$ maximizes the set of small projects that are completed immediately with positive probability. Since at the cutpoint ϕ_1 the single junior agent is indifferent between completing the project and not contributing, we must have $V^1(\phi_1) = 0$. This implies that for all $s > \phi_1$, only the senior member contributes. It is easy to see then that $\sigma^2(s) = s - \phi_1$ for all $s \in (\phi_1, \phi_2]$ with

$$\phi_2 = \phi_1 + \frac{\delta\tilde{\pi}^1}{1-\delta\tilde{\pi}^2}\frac{\theta^2/c}{1-\delta}$$

and $V^2(\phi_2) = 0$. It follows that the project is completed if and only if

$$q \leq \frac{1}{1-\delta}\left[\frac{\theta^1}{c} + \frac{\delta(1-\tilde{\pi}^2)\theta^2}{c(1-\delta\tilde{\pi}^2)}\right], \tag{4.13}$$

and if completed, is completed in at most two steps. The only form of delay and the only form in which free riding manifests itself is inaction.

This result is analogous to the main theorem of Compte and Jehiel (2003). We can think of the above result as extending their finding to the case where recognition among two heterogeneous agents is stochastic instead of sequentially determined. Theorem 4.9 shows that this result depends crucially on there being a single agent of each group, and thus on the non-existence of in-group free-riding incentives.

Next, suppose $n^2 > 1$ and $n^1 = 1$. As in the previous case, here $\phi_1 = (\theta^1/c)/(1-\delta)$ and $V^1(\phi_1) = 0$, and the single type 1 agent is unwilling to contribute for any $s > \phi_1$. The game truncated at final node ϕ_1 now only involves senior agents, who are fully homogeneous, as in Admati and Perry (1991). The equilibrium of the truncated game is characterized by the sequence

$$\phi_\tau = \phi_1 + \frac{\delta\tilde{\pi}^1}{1-\delta\tilde{\pi}^1}\frac{\Delta^2}{(n^2-1)}\left[\frac{1-(\Delta^2)^\tau}{1-\Delta^2} - 1\right]\bar{s}_1 \qquad \text{for } \tau > 1$$

with associated values

$$V^2(\phi_\tau) = (\Delta^2)^{\tau-1}\frac{\tilde{\pi}^1}{1-\delta\tilde{\pi}^2}\theta^2,$$

recovering the structure of the equilibrium in Admati and Perry (1991) with an endogenous number of stints and the presence of gradualism as well as inaction.

The Role of Random Recognition. In team settings, the specific individual with the opportunity to contribute at any given time is often random, as captured in the model.[11] However, one can ask how this property affects the equilibrium. For example, how would the equilibrium structure change if the group-identity of the contributor followed a set pattern, as in Admati and Perry (1991) and Compte and Jehiel (2003)? For illustration, suppose a junior were recognized at random, then a senior, then a junior, and so on.

Let's focus on small projects first. The result of Lemma 4.1 continues to hold: there is a maximum contribution $w^m(1)$ a senior (junior) is willing to make to complete the project, given that a member of the opposite group will complete the project for sure in the next period. In this case, the fact that $\theta^1 \neq \theta^2$ guarantees $w^m(1)$ is different across groups. Suppose seniors have a smaller willingness to complete. Then, for any $s \in (w^2(1), \phi_1]$, seniors will shirk completely. This is because completion in the next period is *guaranteed* via recognition of a junior. Contributing any positive amount that does not complete the project is therefore dominated, while $w^2(1)$ is the maximal amount a senior will contribute to finish.

Like in single agent teams, there is only one step senior agents take within the small project. Random recognition is thus responsible for the infinitely many senior cutpoints below ϕ_1. However, it does not affect the maximal out-group free riding observed in equilibrium at ϕ_1.

Consider now the case of large projects. Recall that at each ϕ_τ, one group was active while the other remained inactive, allowing us to replicate the identity of these groups when the order is predetermined. However, in this case, this structure is inherently dependent on Markovian strategies and does not ensure the uniqueness of the equilibrium. Indeed, it remains uncertain whether all equilibria necessarily exhibit cycles or there are other equilibria without them.

Similarly, one may ask how crucial our assumption of a single recognized agent is for determining the stationarity and uniqueness of equilibrium, as well as the presence of contribution cycles within equilibrium. Indeed, cycles may arise when more than one team member is recognized. However, as in the case of a prespecified order, we cannot rule out the existence of multiple equilibria, since off-equilibrium behavior

[11] We address contribution cycles with multiple recognized members below.

becomes relevant when multiple agents are recognized simultaneously.[12]

At a technical level, random recognition can be conceptualized as introducing effects similar to *partial commitment*, fundamentally altering the strategic environment and the players' behavior. To see this, it is a good idea to compare our model with continuous time public goods contribution games where agents contribute simultaneously and continuously (see Fershtman and Nitzan (1991), Bonatti and Hörner (2011), Marx and Matthews (2000), Bowen et al. (2019), and, in particular, the extension to two groups in Giorgiadis (2015)). In these games, value functions are typically *smooth*, in particular, differentiable, and generally attain an interior maximum at a positive contribution level satisfying first-order conditions.[13]

In contrast, in our model with random recognition, the values of contributions are almost everywhere decreasing and exhibit discontinuities at the cutpoints where they experience upward jumps. As a result, optimal behavior is not characterized by standard first-order conditions. Instead, best responses emerge from non-convexities created by the interaction between the randomness in recognition and the strategic responses in equilibrium. Not surprisingly, our results continue to hold in continuous time when agents are randomly selected but at intervals of length Δ, and cycles disappear when the contributions happen too fast with $\Delta \to 0$.[14]

4.2.2 Efficiency Considerations

We consider two distinct questions related to efficiency: (i) whether all welfare-improving projects are completed, and (ii) what is the delay incurred in finishing efficient projects. Finally, we also compare both the completion of welfare-improving projects and delay to those within a team of *homogeneous* agents.

Our normalization of utility by the factor $1 - \delta$ affects both the project's infinite-lasting flow of payoffs and the one-stage cost of each paid contribution. This normalization simplifies the expressions of the value function as θ^m represents both a flow

[12] When only a single agent is recognized, the best possible deviation does not alter the total contributions. In contrast, with multiple recognized agents, the optimal deviation results in a contribution level that deviates from the equilibrium path. Further details are available upon request.

[13] A similar result holds in some discrete-time settings with simultaneous contributions; see Battaglini et al. (2014).

[14] Additional details are provided in Online Appendix B of the working paper version of this paper.

payoff and the discounted utility of the project. For any $\delta < 1$ this double role of θ^m does not affect the arguments above and, in particular, does not affect the efficiency considerations. However, as $\delta \to 1$, any finite contribution in a single period vanishes relative to the discounted utility of the infinite-lasting project. Hence, when discounting disappears, efficiency considerations are more subtle. To simplify the expressions in this section, we consider a project that, once completed, lasts for T finite periods and delivers θ^m in each period. The utility of such a finished project to agent i of group $m = 1, 2$ is

$$U^i = (1-\delta)\sum_{t=1}^{T} \delta^{t-1}\theta^m = (1-\delta)\hat{\theta}^m \quad \text{where} \quad \hat{\theta}^m \equiv \left(\frac{1-\delta^T}{1-\delta}\right)\theta^m.$$

Implementable Projects. By expression (4.7) in Theorem 4.5, for any $\tau > 1$, if group m is the unique contributor at τ,

$$\phi_\tau - \phi_{\tau-1} = \left(\frac{1}{1-\delta+\delta\tilde{\pi}^m \frac{n^m-1}{n^m}}\right)\frac{\delta V^m(\phi_{\tau-1})}{c}.$$

where

$$V^m(\phi_{\tau-1}) = \begin{cases} \alpha^m V^m(\phi_{\tau-2}) & \text{if } m_{\tau-1} = m \\ \beta^m V^m(\phi_{\tau-2}) & \text{if } m_{\tau-1} \neq m \end{cases}$$

Since $\alpha^m < \beta^m < \delta$, the value $V^m(\phi_\tau)$ is decreasing in τ, and goes to zero as $\tau \to \infty$. It follows that the incremental contribution $\phi_\tau - \phi_{\tau-1}$ is decreasing in τ and goes to zero as $\tau \to \infty$. This means that the sequence $\{\phi_\tau\}$ converges to a point $\tilde{\phi}$. The threshold $\tilde{\phi}$ gives the size of the smallest project that will not be completed in equilibrium. Using (4.7) and the derived values for each agent at the end of any given contribution stint, we can calculate the cutoff associated with the last step of stint k, $\phi_{J(k)}$, for any $k \geq 1$. Proposition 4.8 then allows us to obtain an upper bound for the limit $\tilde{\phi}$.[15] Using this bound, we address the question of whether all projects which would be efficient to pursue – i.e., projects such that $cq < (n^1\hat{\theta}^1 + n^2\hat{\theta}^2)$ – are in fact completed. We call these *efficient projects* for short. We show that not all efficient projects are completed, even in the limit as $\delta \to 1$.

Theorem 4.9. *For any $\delta < 1$, $c\tilde{\phi} < \hat{\theta}^1 + \hat{\theta}^2$. Hence, (i) for any fixed $\delta < 1$, some*

[15] We present the exact characterization of this bound in the proof of Theorem 4.9.

efficient projects are not completed. Moreover, (ii) if $n^m > 1$ for any $m \in \{1,2\}$, then even in the limit as $\delta \to 1$, some efficient projects are not completed.

The qualification that $n^m > 1$ for any $m \in \{1,2\}$ is important. As we discussed in the previous section, when $n^1 = n^2 = 1$ the project is completed if and only if (4.13) holds, and if completed, it is completed in at most two steps. It follows that

$$c\tilde{\phi}_{n^1=n^2=1} = \hat{\theta}^1 + \frac{1-\tilde{\pi}^2}{1-\delta\tilde{\pi}^2}\delta\hat{\theta}^2$$

which implies that in the limit as $\delta \to 1$, all efficient projects are completed. We thus see that as agents become arbitrarily patient, *in-group* free riding is uniquely responsible for inefficiency in project completion.[16]

Delay. As we have shown before, the incremental contribution $\phi_\tau - \phi_{\tau-1}$ in (4.7), is decreasing in τ and goes to zero as the sequence $\{\phi_\tau\}$ approaches the limit $\tilde{\phi}$. It follows that (i) the number of steps required to finish a project of size q, $J(\hat{k}(q))$, where $\hat{k}(q) \equiv \min\{k : \phi_{J(k)} \geq q\}$, is an increasing and convex function of q, and (ii) $J(\hat{k}(q)) \to \infty$ as $q \to \tilde{\phi}$. This means that the number of contribution stints required to complete a project of size q, $\hat{k}(q)$, is also an increasing and convex function of q, such that $\hat{k}(q) \to \infty$ as $q \to \tilde{\phi}$. In our next result, we use the structure of equilibrium strategies characterized in the previous section to compute the expected delay $\mathcal{E}(q)$ associated with completion of a project of size q.

Proposition 4.10. *Suppose $q = \phi_\ell$ for $\ell > j(1)$, and let $\mathcal{E}_1 \equiv j(1) + \tilde{\pi}^2(\tilde{\pi}^1)^{j(1)-2}$. If $\Delta^m \geq \Delta^{m'}$, the expected delay associated with completion of a project of size q is:*

$$\tilde{\mathcal{E}}(q) = \mathcal{E}_1 + \left(\frac{\hat{k}(q)-1}{2}\right)\left\{(1+x) + \frac{\tilde{\pi}^m}{\tilde{\pi}^{m'}} + \tilde{\pi}^{m'}(\tilde{\pi}^m)^{x-2}\right\}. \tag{4.14}$$

for some $x \in [x^m, x^m+1]$, where $x^m \equiv \left\lfloor \log[\Delta^{m'}]/\log[\Delta^m]\right\rfloor$.[17] *In particular, $\tilde{\mathcal{E}}(\cdot)$ is an increasing and convex function, and $\lim_{q\to\tilde{\phi}} \tilde{\mathcal{E}}(q) = \infty$.*

[16] Battaglini et al. (2014) obtain efficiency in a model with a single group of arbitrary size. Their setup differs from ours in two important respects: public goods are not binary, and agents can also allocate effort to a private good, which helps smooth investment in the public good. In our setup, random recognition creates opportunities to free ride on members of the other group. However, these opportunities vanish as delay becomes costless, i.e., as agents become arbitrarily patient.

[17] We state the result for an odd number of contribution stints. In the proof of Proposition 4.10, we

Together with the characterization for delay in small projects given by (4.6), Proposition 4.10 gives a full characterization of delay for arbitrary project length q. The result shows that the expected delay increases non-linearly with the size of the project, and goes to infinity as the size of the project approaches the smallest project not completed in equilibrium, $\tilde{\phi}$.

4.2.3 Comparison with homogeneous Teams.

We compare the performance of a team in which $2n$ agents have an identical value of θ for completing the project (*homogeneous team*), to a *two-group team* in which there are two payoff-distinct groups of equal size n: group 1 members obtain payoff $\theta+\varepsilon$ and group 2 members $\theta-\varepsilon$, for some $\varepsilon \in [0,\theta]$, upon completion. We assume all agents are equally likely to be recognized to contribute, and have per-unit contribution cost c, making the set of welfare-enhancing projects the same across teams.[18]

Implementable Projects. We begin by noting that equilibrium behavior within the homogeneous set of agents is akin to that of a two-group team (each group of size $2n$) in which the recognition probability of the outside group vanishes ($\tilde{\pi}^2 = 0$). Using Theorem 4.9, we obtain the largest project implementable by the homogeneous team of $2n$ members: $\tilde{\phi}_{2n} = \theta/(c(1-\delta))$. On the other hand, our previous analysis directly allows us to obtain the maximum project size $\tilde{\phi}$ undertaken by the two-group team when both teams have the same in and out-group free-riding incentives; i.e., $n^1 = n^2$ and $\tilde{\pi}^1 = \tilde{\pi}^2$. From Theorem 4.5, the values of the members of each group at the start of the final small project are given by $\delta V^1(\phi_1) = \alpha \cdot (\theta+\varepsilon)$ and $\delta V^2(\phi_1) = \beta \cdot (\theta-\varepsilon)$, respectively, where

$$\alpha = \frac{\delta(n-1)}{2n-\delta(n+1)}, \qquad \beta = \frac{\delta}{2-\delta}.$$

provide the result for the case of an even number of contribution stints. Note also that the expression of expected delay in the proposition assumes that the project starts exactly at a cutpoint, i.e., $q = \phi_\ell$ for some $\ell > j(1)$. It is straightforward to compute expected delay when the project starts at a point interior to a small project, using the expression (4.6) for expected delay in small projects.

[18] As in the rest of the paper, here we restrict attention to symmetric SPE. A possible question is whether we could support the equilibrium behavior of the two-group case within a homogeneous team, by arbitrarily dividing members into two equal-sized groups. This is in fact a valid equilibrium construction; *it is, however, not robust.* In particular, when costs deviate even an arbitrarily small degree from linearity to become convex, an equilibrium of this form no longer exists. On the other hand, the equilibrium we consider for homogeneous agents is robust to this perturbation. The same is true for the equilibrium described for the two-group case throughout the paper, where groups are distinguished by tangible heterogeneity in valuations or recognition probabilities.

Furthermore, on the equilibrium path we obtain a global one-to-one alternation in group responsibility over small projects, whenever $\varepsilon < [(1-\delta)/((2-\delta)n-1)]\,\theta$, which we assume holds. Using (4.7) and (4.9) recursively, we obtain

$$\phi_k = \frac{2n}{c(2n-\delta\,(n+1))}\left[(\theta+\varepsilon)\sum_{\ell=0}^{\lfloor\frac{k-1}{2}\rfloor}(\alpha\beta)^{\ell} + \beta\cdot(\theta-\varepsilon)\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor-1}(\alpha\beta)^{\ell}\right],$$

with

$$\lim_{k\to\infty}\phi_k \equiv \tilde{\phi} = \frac{2n}{c(2n-\delta\,(n+1))}\left(\frac{(1+\beta)\theta+(1-\beta)\varepsilon}{1-\alpha\beta}\right).$$

We see from above that, within the prescribed range for ε, the size of the largest implementable project is *increasing* in inter-group heterogeneity ε. Moreover, since

$$\lim_{\epsilon\to 0}\tilde{\phi} > \frac{\theta}{c(1-\delta)} = \tilde{\phi}_{2n}$$

we conclude that the introduction of within-team heterogeneity improves the size of the largest implementable project by an amount bounded away from 0, for any fixed δ, n. As n grows large, this benefit decreases and becomes zero in the limit.

Delay. We now compare project completion times across the two team structures, focusing again on the case where $\varepsilon \searrow 0$. The analysis we have done so far is nearly sufficient to undertake this comparison. Suppose $q = \phi_k$, as in the characterization of delay given by Proposition 4.10. In the two-group team, the expected completion time for a project of size q is exactly $2k$ periods, since each small project takes *two* periods to complete in expectation. On the other hand, the homogeneous team moves precisely one cutpoint in each period. This highlights a trade-off: the homogeneous team is always able to move the project forward, but in small steps. In contrast, in the two-group team, out-group free-riding causes periods of inaction, but each contribution is larger. The resolution of this trade-off is not trivial but, for sufficiently large projects, the two-group team always dominates in performance:

Proposition 4.11. *Comparing the two-group and homogeneous team performance,*

(i) The two-group team is able to complete larger projects than the homogeneous team. Moreover, as $\varepsilon \to 0$ (i.e., intergroup differences vanish in the two-group team), the difference in the size of the maximal completable projects by each

team remains bounded away from zero, for any δ, n.

(ii) Whenever the project is sufficiently large (i.e., if $q = \phi_k$, there exists $k^*(\delta, n) \in \mathbb{N}$ such that whenever $k \geq k^*(\delta, n)$), the two-group team completes it in a shorter time in expectation.

Our findings reveal a surprising recommendation for homogeneous groups tackling projects that seem too large to complete. Managers can design small rewards that split the group into two equally sized subgroups, creating artificial heterogeneity. This slight variation in preferences both accelerates completion and makes otherwise unattainable projects feasible by reducing free riding. This approach reflects the beneficial heterogeneity emphasized by Olson's insight quoted in the introduction.

5 Conclusions

In this paper, we study inter-group collaboration within a team of agents using a model of sequential contributions to a joint project in a decentralized environment. We assume the joint project is a *binary public good*, delivering benefits inexcludably to the team of agents only once completed, and that there is no deadline or asymmetric information among agents. Crucially, we focus on a team naturally divided into two sub-groups, each of which has a distinct value from completing the project.

We show that in large projects, the equilibrium of the model displays endogenous contribution cycles, in which agents from different groups within the team alternate making gradual contributions towards project completion. Importantly, *delay*, *inaction*, and *alternation* in work between the sub-groups — familiar features of the teamwork setting— all emerge as features of equilibrium. Neither collaboration between homogeneous, nor fully heterogeneous agents captures these stylized facts. This points to the salience of the *inter-group* case as a descriptive framework for collaborative settings whenever contributions are transparent and decentralized.

Throughout the paper, we focus on binary public goods/joint projects. In other cases, agents may *also* obtain a flow payoff even if the project is incomplete; i.e., $u^m(s(t)) = r^m(q - s(t))$ for $s(t) > 0$ and $u^m(0) = qr^m + \theta^m > 0$. Our analysis extends to this case with suitable modifications when the flow payoff r^m is "small."

Thus, our conclusions can be extended to applications in which the terminal payoff dominates. When the flow payoff r^m is large relative to terminal payoffs θ^m, however (e.g., collaboration across countries to reduce carbon emissions), the strategic analysis changes qualitatively. We believe that extending the analysis in this paper to the continuous public good case would be a promising avenue for future research.

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A Appendix A: Proofs

Proof of Lemma 4.1

(a) First we show that $\exists b > 0$ such that $\sigma^m(h) = s^h$ for all m and all histories h such that $s^h \leq b$. Note that for $s^h \leq \min\{\theta^1, \theta^2\}/c$,

$$W^m(s^h, h) = \theta^m - (1-\delta)c \cdot s^h \geq \delta\theta^m, \ \forall m,$$

where the right-hand side is an upper bound on $W^m(e, h)$ for any $e < s^h$, since it is the value from zero contribution and the project being completed by another agent in the subsequent period. Letting $b = \min\{\theta^1, \theta^2\}/c$ completes the step.

(b) We now determine $w(1) \equiv \sup\{b : \sigma^m(s) = s \ \forall s \leq b, \forall m \in M\}$. Note that from any h, that for $e < s^h$: $W^m(e, h) = \delta V^m(h_e) - (1-\delta)ce$. Take any h with $s^h \leq w(1)$. From (3.3), and since $\sigma^m(h') = s(h')$ for all m for histories h' with $s(h') < w(1)$, for any $e < s^h$, $V^m(h_e) = \theta^m - \frac{\tilde{\pi}^m}{n^m} c(1-\delta)(s^h - e)$. Thus for $e < s^h$:

$$W^m(e, h) = \delta\theta^m - c(1-\delta)\left(e + \delta\frac{\tilde{\pi}^m}{n^m}(s^h - e)\right).$$

Note that for $e \in (0, s^h)$, $s^h < w(1)$, $W^m(0, h) - W^m(e, h) > 0$. Thus, when all agents finish the project on their turn, an agent is better off contributing zero than contributing $e \in (0, s^h)$. Moreover, $W^m(s^h, h) \geq W^m(0, h)$ if and only if

$$s^h \leq \frac{\theta^m}{c\left(1 - \delta\frac{\tilde{\pi}^m}{n^m}\right)} \equiv w^m(1),$$

with equality uniquely at $s = w^m(1)$. Thus $\sigma^m(h) = s^h$ for $m = 1, 2$ if $s < w(1) \equiv \min_m w^m(1)$ and only if $s \leq w(1)$. In particular, from any history h with $s^h = w(1)$, we see it is always an equilibrium that $\sigma^m(h) = w(1)$ for $m = 1, 2$.

(c) We show that equilibrium behavior at any history with corresponding state $w(1)$ is unique and entails completion of the project. Note that if $w^m(1) > w(1)$, group m agents have a strict incentive to contribute at such an h, so we need only consider m for which $w^m(1) = w(1)$. From previous work, we know that $\sigma^m(h) \in \{0, w(1)\}$. Consider therefore an equilibrium in which $\sigma^m(h) = 0$. In such an equilibrium, $V^m(h) = \frac{\tilde{\pi}^{m'}\theta^m}{1-\delta\tilde{\pi}^m}$. A deviation to contributing $w(1)$ at h is profitable, since

$$W^m(w(1), h) = \delta\theta^m \frac{n^m - \tilde{\pi}^m}{n^m - \delta\tilde{\pi}^m} = \delta V^m(h)\frac{\frac{n^m-\tilde{\pi}^m}{n^m-\delta\tilde{\pi}^m}}{\frac{1-\tilde{\pi}^m}{1-\delta\tilde{\pi}^m}} > \delta V^m(w(1)) \tag{A.1}$$

(d) It remains to show that if $w^m(1) > w^{m'}(1)$, then $\sigma^m(h) = s^h$ for all h with $s^h \in (w^{m'}(1), w^m(1)]$. Consider first $s^h \in (w^{m'}(1), w^m(1))$. As before, $W^m(s^h, h) = \theta^m - c(1-\delta)s^h$. Since group m' agents don't finish outright, for any $e \in [0, s^h)$,

$$W^m(e, h) < \delta\theta^m - c(1-\delta)\left[e + \delta\frac{\tilde{\pi}^m}{n^m}(s^h - e)\right] \leq \delta\left(\theta^m - c(1-\delta)\frac{\tilde{\pi}^m}{n^m}s^h\right)$$

Thus $W^m(s^h, h) > W^m(e, h)$ for all $e \in [0, s^h)$ if and only if $s^h < w^m(1)$. Now consider a history h for which $s^h = w^m(1)$. Using similar arguments we have that $W^m(w^m(1), h) > W^m(e, h)$ for all $e \in (0, w^m(1))$, so we only need to consider equilibria with $\sigma^m(h) \in \{0, w^m(1)\}$. The definition of $w^m(1)$ gives that $\sigma^m(h) = w^m(1)$ is indeed an equilibrium. For uniqueness, assume that $\sigma^m(h) = 0$ from some history h with corresponding state $w^m(1)$. It follows then that

$$V^m(h) = \frac{\tilde{\pi}^{m'}\delta}{1 - \tilde{\pi}^m\delta}V^m\left(h_{\sigma^{m'}(h)})\right)$$

Since no type finishes the project outright, we must have that

$$V^m\left(h_{\sigma^{m'}(h)}\right) \leq \theta^m - \frac{\tilde{\pi}^m}{n^m}c(1-\delta)\left(w^m(1) - \sigma^{m'}(h)\right)$$

Therefore, there is a profitable deviation for type m if $\delta V^m(h) < \theta^m - c(1-\delta)w^m(1)$, which holds, since $\frac{1-\tilde{\pi}^m}{n^m-\tilde{\pi}^m}\sigma^{m'}(h) < w^m(1)$.

□

Proof of Proposition 4.2

We prove this in a series of steps.

Step 1. *Suppose $s_1(2) > w(1)$ is such that for all histories h with $s^h \leq s_1(2)$, $\sigma^1(h) = s^h$. Then:*

1. For any h with corresponding state $s^h \in (s_1(1), s_1(2)]$, $\sigma^2(h) = s^h - w(1)$.

2. For any h with $s^h > s_1(2)$, $\sigma^2(h) \leq s^h - s_1(2)$.

Proof. Suppose h is such that $s^h \in (s_1(1), s_1(2)]$, and consider the problem of a senior (i.e., group 2) agent, i. We have shown that for any history h' with $s^{h'} > w(1)$, $\sigma^2(h') < s^{h'}$. Contributing $e \in (s^h - w(1), s^h)$ from h is clearly dominated, since by moving the state to $w(1)$, the project will be finished at the same time, with lower expected costs. Thus, $\sigma^2(h) \in [0, s^h - w(1)]$. Suppose first $\sigma^2(h) = s^h - w(1)$ for all histories h with $s^h \in (s_1(1), s_1(2)]$. Note that since all agents finish the project outright for any history h' with $s^{h'} \leq w(1)$, if agent i contributes $e = s^h - w(1)$, from the history h, she gets a payoff

$$\begin{aligned} W^2(s^h - w(1), h) &= \delta V^2(h_{s^h - w(1)}) - (1-\delta)c(s^h - w(1)) \\ &= \delta \left[\theta^2 - \frac{\tilde{\pi}^2}{n^2} c(1-\delta)w(1) \right] - (1-\delta)c(s^h - w(1)). \end{aligned} \tag{A.2}$$

Now suppose i deviates and contributes $e \in [0, s^h - w(1))$, say $e = s^h - w(1) - \varepsilon$, with $0 < \varepsilon \leq s^h - w(1)$. This gives her a payoff

$$\begin{aligned} W^2(e,h) &= \delta V^2(h_e) - (1-\delta)c(s^h - w(1) - \varepsilon) \\ &= \delta\theta^2(\tilde{\pi}^1 + \delta\tilde{\pi}^2) - (\delta)^2\tilde{\pi}^2(1-\delta)\frac{\tilde{\pi}^2}{n^2}cw(1) - (1-\delta)c(s - w(1)) + \varepsilon(1-\delta)c(1 - \frac{\tilde{\pi}^2}{n^2}\delta), \end{aligned} \tag{A.3}$$

Note that this expression is increasing in ε. Thus, the binding deviation is $e = 0$, or $\varepsilon = s^h - w(1)$. This is not a profitable deviation iff

$$s^h \leq \left(\frac{\delta\tilde{\pi}^2}{1 - \delta\frac{\tilde{\pi}^2}{n^2}} \right) \frac{\theta^2}{c} + \left\{ \frac{1 - \delta\frac{\tilde{\pi}^2}{n^2}(2 - \delta\tilde{\pi}^2)}{1 - \delta\frac{\tilde{\pi}^2}{n^2}} \right\} w(1) = w(1) + w(1) \left[\frac{\delta\frac{\tilde{\pi}^2}{n^2}(n^2 - 1)}{1 - \delta\frac{\tilde{\pi}^2}{n^2}} \right] = s_1(2).$$

Suppose instead that in equilibrium $\sigma^2(h) < s^h - w(1)$ for some history h with $s^h \in (s_1(1), s_1(2)]$. Note that the equilibrium payoff for a senior agent at h is bounded above by the RHS of A.3. A deviation to $e = s^h - w(1)$ at h gives a payoff A.2. Since

$s^h \in (s_1(1), s_1(2)]$, it follows that this is a profitable deviation, and thus in equilibrium we must have $\sigma^2(h) = s^h - w(1)$ for all histories h with $s^h \in (s_1(1), s_1(2)]$.

For part 2, note that our previous argument shows that $\sigma^2(h) < s^h - w(1)$ for all h with $s^h > s_1(2)$. Now, for any such h, moving the project to any history with corresponding state $s \in (s_1(2), w_1)$ is clearly dominated by moving the project to a history with corresponding state $s_1(2)$, since the project will be finished at the same time, with lower expected costs. Thus $\sigma^2(h) \leq s^h - s_1(2)$ for all $s^h > s_1(2)$.

By definition in footnote 7, when senior agents play a cutpoint strategy in $[0, b]$, their strategy is *history-independent* from all histories h for which $s^h \leq b$. We thus use the convention of writing $\sigma^m(s)$ in place of $\sigma^m(h)$, and $V^m(s)$ in place of $V^m(h)$ for any history h such that $s^h = s$, whenever group-m agents play a cutpoint strategy.

Step 2. *Suppose there is $b \in \mathbb{R}_+$ such that senior (group 2) agents follow a cutpoint strategy in $[0, b]$ and $\sigma^1(h) = s^h$ at all h for which $s^h \leq b$. Then (1) the sequence of cutpoints is described by (4.2), and (2) the values for junior (group 1) agents are given by*

$$V^1(s_1(1)) = \theta^1 - \tilde{\pi}^1 c(1-\delta) s_1(1), \tag{A.4}$$

and for $k > 1$,

$$\begin{aligned} V^1(s_1(k)) = &\left(\frac{1-(\tilde{\pi}^2\delta)^k}{1-\tilde{\pi}^2\delta}\right)\left[\tilde{\pi}^1\theta^1\right] + (\tilde{\pi}^2\delta)^k\theta^1 \\ &- \frac{\tilde{\pi}^1}{n^1}\left(\frac{1-\delta}{1-\delta\tilde{\pi}^2}\right)\left[\left(\frac{1-(\delta\tilde{\pi}^2)^k}{1-\delta\tilde{\pi}^2}\right) - \left(\delta\tilde{\pi}^2\right)^k\left(\frac{n^2-\delta\tilde{\pi}^2}{1-\delta\tilde{\pi}^2}\right)\left[1-\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^k\right]\right]\theta^2 \end{aligned} \tag{A.5}$$

Proof. Using (4.3) for $s_1(k)$ and $s_1(k-1)$ in (3.3b) gives

$$s_1(k) = w_2(1) + \delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)s_1(k-1) \tag{A.6}$$

Solving the difference equation, we have:

$$s_1(k) = \left[\sum_{\ell=0}^{k-1}\left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^\ell\right]\left[\frac{1}{c}\left(\frac{n^2}{n^2-\delta\tilde{\pi}^2}\right)\theta^2\right]$$

and using that $\sum_{\ell=0}^{k-1} x^\ell = \frac{1-x^k}{1-x}$ we get (4.2). The values of junior agents for $k = 1$

follows trivially, as agents in both groups contribute to complete the project at $s_1(1)$. Consider $k > 1$. The value for a junior at $s_1(k)$ is

$$V^1(s_1(k)) = \tilde{\pi}^1\theta^1 + \tilde{\pi}^2\delta V^1(s_1(k-1)) - \frac{\tilde{\pi}^1}{n^1}c(1-\delta)s_1(k), \qquad \text{and} \tag{A.7}$$

$$V^1(s_1(k-1)) = \tilde{\pi}^1\theta^1 + \tilde{\pi}^2\delta V^1(s_1(k-2)) - \frac{\tilde{\pi}^1}{n^1}c(1-\delta)s_1(k-1)$$

So substituting, recursively,

$$V^1(s_1(k)) = \left(\frac{1-(\tilde{\pi}^2\delta)^{k-1}}{1-\tilde{\pi}^2\delta}\right)\tilde{\pi}^1\theta^1 + (\tilde{\pi}^2\delta)^{k-1}\theta^1 - \frac{\tilde{\pi}^1}{n^1}c(1-\delta)\left[\sum_{\ell=0}^{k-1}(\tilde{\pi}^2\delta)^\ell s_1(k-\ell)\right].$$

Substituting $s_1(k-\ell)$, and simplifying, step 2 follows.

Step 3. *In any symmetric SPE, σ^2 is a cutpoint strategy in $[0, \overline{s}_1]$ for $\overline{s}_1 \equiv \sup_{k\geq 1} s_1(k)$ if $\sigma^1(h) = s^h$ for all histories h with $s^h \leq \overline{s}_1$.*

Proof. We proceed by induction. First, by lemma 4.1 and step 1 above, the statement is true up to $k = 2$, with $s_1(0) \equiv 0$, $s_1(1) \equiv w(1)$, and $s_1(2)$ as defined in step 1. Next, suppose the statement is true up to k, and consider any history h with $s^h \in (s_1(k), s_1(k+1)]$. Note that if for any such h, $\sigma^2(h) \leq s^h - s_1(k)$, an argument analogous to that of step 1 establishes the induction step. To prove this, we show that for any history h with $s^h \in (s_1(k), s_1(k+1)]$, if $\sigma^2(h) > s^h - s_1(k)$, a senior (group 2) agent would gain by deviating to contributing zero. Since moving to a non-cutpoint is dominated by moving to a cutpoint, it suffices to consider $\sigma^2(h) = s^h - s_1(k-\ell)$ $\forall$ $\ell : 1 \leq \ell < k$. In the proposed equilibrium, a senior that gets to contribute has payoff

$$W^2(s^h - s_1(k-\ell), h) = \delta V^2(s_1(k-\ell)) - (1-\delta)c(s^h - s_1(k-\ell)) = \theta^2 - (1-\delta)cs^h$$

A deviation to zero gives $W^2(0, h) = \delta V^2(h)$. Since by assumption $\sigma^2(h) = s^h - s_1(k-\ell)$ and we know $\sigma^1(h') = s^{h'}$ for all histories with $s^{h'} \leq \overline{s}_1$, at s we have

$$V^2(h) = \tilde{\pi}^2\delta V^2(s_1(k-\ell)) + (1-\tilde{\pi}^2)\theta^2 - \frac{\tilde{\pi}^2}{n^2}c(1-\delta)[s^h - s_1(k-\ell)].$$

$$\Longrightarrow W^2(0, h) = \delta\theta^2 - \delta\tilde{\pi}^2(1-\delta)cs_1(k-\ell) - \delta\frac{\tilde{\pi}^2}{n^2}c(1-\delta)[s^h - s_1(k-\ell)]$$

Thus, contributing zero is a profitable deviation iff

$$\left(\frac{n^2-\delta\tilde{\pi}^2}{n^2}\right)s^h-\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2}\right)s_1(k-\ell)>\frac{\theta^2}{c}$$

Note that if this is satisfied for $\ell=1$, it is satisfied for all $\ell:1\leq\ell<k$. Moreover, recall that $s^h>s_1(k)$, so writing $s^h=s_1(k)+\varepsilon$, for $\varepsilon>0$, this is profitable iff

$$\left(\frac{n^2-\delta\tilde{\pi}^2}{n^2}\right)(s_1(k)+\varepsilon)-\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2}\right)s_1(k-1)>\frac{\theta^2}{c}$$

and using (A.6), the inequality becomes $\left(\frac{n^2-\delta\tilde{\pi}^2}{n^2}\right)\varepsilon>0$, which always holds.

Having shown that for any history h with $s^h\in(s_1(k),s_1(k+1)]$, $\sigma^2(h)\leq s^h-s_1(k)$, an argument analogous to that of step 1 shows that in equilibrium $\sigma^2(h)=s^h-s_1(k)$ for any h with $s^h\in(s_1(k),s_1(k+1)]$. This concludes the proof of step 3.

Step 4. *Suppose $\overline{s}_1<\phi_1$ and $\sigma^1(h)=s^h$ from any h with $s^h\leq\phi_1$. Then $\sigma^2(h)=0$ from all histories h with $s^h\in[\overline{s}_1,\phi_1]$.*

Proof. (i) First, note that for all h with $s^h\in[\overline{s}_1,\phi_1]$, $\sigma^2(h)<s^h-s_1(k)$ for all k in the sequence. The formal argument is the same the proof of step 3, and therefore omitted here. (ii) Now suppose for $s^h\in[\overline{s}_1,\phi_1]$, $\sigma^2(h)\in[0,s^h-\overline{s}_1]$, say $\sigma^2(h)=s^h-\overline{s}_1-\varepsilon$, with $0\leq\varepsilon\leq s^h-\overline{s}_1$. This gives a group-2 agent payoff

$$W^2(e,h)=\delta V^2(h_e)-(1-\delta)c(s^h-\overline{s}_1-\varepsilon)$$

Now, by part (i), any contribution $e\in[0,s^h-\overline{s}_1]$ will never result in a move to a state in which a senior agent advances the project to any $s_1(k)$ in the sequence. Thus $V^2(h_e)\leq(1-\tilde{\pi}^2)\theta^2$. Hence, for any $e=s-\overline{s}_1-\varepsilon$, with $0\leq\varepsilon\leq s-\overline{s}_1$, $W^2(e,h)\leq\delta(1-\tilde{\pi}^2)\theta^2-(1-\delta)c(s^h-\overline{s}_1-\varepsilon)$. It follows that if $\sigma^2(h)>0$ for any such h, a senior (group 2) agent would gain by deviating to contributing zero. To complete the proof, we show that if $\sigma^2(h)=0$ for all h with $s^h\in[\overline{s}_1,s_1(1)]$, seniors don't have a profitable deviation in this interval. For any such h:

$$V^2(h)=\left(\frac{1-\tilde{\pi}^2}{1-\delta\tilde{\pi}^2}\right)\theta^2.$$

A deviation to any cutpoint $s_1(k)$ gives

$$W^2(s^h - s_1(k), h) = \theta^2 - (1-\delta)cs_1(k) - c(1-\delta)(s - s_1(k)) = \theta^2 - (1-\delta)cs$$

where we used (4.3). This is a profitable deviation iff $W^2(s^h - s_1(k), s) > \delta V^2(h)$ or

$$s^h < \left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c} = \overline{s}_1.$$

Step 5. *In any equilibrium, there are infinitely many cupoints $\{s_1(k)\}$ below ϕ_1.*

Proof. Assume, towards a contradiction, that $\exists\ \hat{k} < \infty$ such that $s_1(\hat{k}) < \phi_1(\hat{k}) \leq s_1(\hat{k}+1)$. Recall that, by definition, $\phi_1(\hat{k})$ coincides with ϕ_1 when there are exactly $\hat{k}$ cutpoints of juniors below ϕ_1. The assumed condition is thus necessary to sustain an equilibrium with exactly $\hat{k}$ senior cutpoints. The above steps show that seniors play a history-independent strategy from any h with $s^h < \phi_1$; moreover, juniors complete outright from any h with $s^h \leq \phi_1$. So, we use the notation $\sigma^m(s)$ and $V^m(s)$, again, to mean the strategy and value of group-m agents from any h with $s^h = s$.

(a) Note that if $s_1(\hat{k}) < \phi_1(\hat{k}) \leq s_1(\hat{k}+1)$, it must be that $\sigma^2(s) = s - s_1(\hat{k})$ and $\sigma^1(s) = s$ for all $s \in (s_1(k), \phi_1(\hat{k})]$, so any junior agent's value at $\phi_1(\hat{k})$ is:

$$V^1\phi_1(\hat{k})) = \tilde{\pi}^1\theta^1 - \frac{\tilde{\pi}^1}{n^1}c(1-\delta)\phi_1(\hat{k}) + \tilde{\pi}^2\delta V^1(s_1(\hat{k})) \tag{A.8}$$

where $V^1(\phi_1(\hat{k}))$ must verify $\delta V^1(\phi_1(\hat{k})) = \theta^1 - c(1-\delta)\tilde{s}(1,\hat{k})$. Replacing in (A.8),

$$\phi_1(\hat{k}) = w^1(1)\left[1 + \frac{\delta\tilde{\pi}^2}{1-\delta}\frac{\theta^1 - \delta V^1(s_1(\hat{k}))}{\theta^1}\right] \tag{A.9}$$

(b) Note that if $s_1(\hat{k}+1) < \phi_1(\hat{k})$ there is no equilibrium with $\hat{k}$ cut-points for seniors below $\phi_1(\hat{k})$. This is because while junior agents contribute to complete the project, the maximal amount any senior agent is willing to contribute to obtain the state $s_1(\hat{k})$ is $s_1(\hat{k}+1) - s_1(\hat{k}) < \phi_1(\hat{k}) - s_1(\hat{k})$. Thus, at $\phi_1(\hat{k})$, only a junior agent would be willing to contribute, contradicting the hypothesized equilibrium structure.

We now show that $\phi_1(k) > s_1(k+1)\ \forall k \geq 1$. Consider first $k \geq 2$. From part (a),

$$\begin{aligned}\phi_1(k) &= \left[\frac{n^1}{n^1 - \delta\tilde{\pi}^1}\right]\frac{1}{c}\left\{\left[\frac{1-\delta\tilde{\pi}^1}{1-\delta}\right]\theta^1 - \left(\frac{\delta^2}{1-\delta}\right)\tilde{\pi}^2 V^1(s_1(k))\right\} \\ &= \frac{n^1}{n^1-\delta\tilde{\pi}^1}\cdot\frac{1}{c}\left\{\theta^1\cdot\left[\frac{1-(\delta\tilde{\pi}^2)^{k+1}}{1-\delta\tilde{\pi}^2}\right] + \theta^2 T(n^2)\right\}\end{aligned}$$

where

$$T(n^2) \equiv \frac{\delta^2\tilde{\pi}^1\tilde{\pi}^2\left(1-(\delta\tilde{\pi}^2)^k\left(n^2-(n^2-1)\left(1-\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^k\right)\right)\right)}{n^1(1-\delta\tilde{\pi}^2)^2}$$

results from substituting $V^1(s_1(k))$ from (A.5) and simplifying. Noting that

$$\frac{\partial}{\partial n^2}\left(n^2-(n^2-1)\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^k\right) = 1 + k\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^{k-1} - (k+1)\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^k,$$

and that for any k, the function $f(x) = k\cdot x^{k-1} - (k+1)x^k$ has its maximum at $x = \frac{k-1}{k+1}$ with $f(\frac{k-1}{k+1}) = \left(\frac{k-1}{k+1}\right)^{k-1}$, we see that the LHS of this equation is positive. Moreover, since the limit as $n^2 \to \infty$ of the argument in the LHS is $(1-\delta\tilde{\pi}^2)k$,

$$T(n^2) > \lim_{n^2\to\infty} T(n^2) = \frac{\delta^2\tilde{\pi}^1\tilde{\pi}^2\left(1-(\delta\tilde{\pi}^2)^k\left(1+(1-\delta\tilde{\pi}^2)\cdot k\right)\right)}{n^1(1-\delta\tilde{\pi}^2)^2} > 0 \quad \forall k \geq 2.$$

Therefore, for any $k \geq 2$,

$$\frac{\phi_1(k)}{s_1(k+1)} > \frac{n^1}{n^1-\delta\tilde{\pi}^1}\cdot\frac{\theta^1}{\theta^2}\cdot\frac{1-(\delta\tilde{\pi}^2)^{k+1}}{1-\left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^{k+1}}.$$

Further using the fact that $w^1(1) > w^2(1) \implies \frac{n^1}{n^1-\delta\tilde{\pi}^1}\cdot\frac{\theta^1}{\theta^2} > \frac{n^2}{n^2-\delta\tilde{\pi}^1}$, we obtain

$$\begin{aligned}\frac{\phi_1(k)}{s_1(k+1)} &> \frac{n^2}{n^2-\delta\tilde{\pi}^2}\cdot\frac{1-(\delta\tilde{\pi}^2)^{k+1}}{1-\left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^{k+1}} \\ &= \left(1-(\delta\tilde{\pi}^2)^{k+1}\right)\cdot\left[\frac{n^2(n^2-\delta\tilde{\pi}^2)^k}{(n^2-\delta\tilde{\pi}^2)^{k+1}-(\delta\tilde{\pi}^2)^{k+1}(n^2-1)^{k+1}}\right] := R(n^2,k).\end{aligned}$$

For any fixed, arbitrary k, $\frac{\partial}{\partial n^2}R(n^2,k) < 0$ so $R(n^2,k) > \lim_{n^2\to\infty} R(n^2,k) = 1$ and hence, from above, $\frac{\phi_1(k)}{s_1(k+1)} > 1$, which yields the desired result $\phi_1(k) > \psi(k+1)$, for any $k \geq 2$.

Consider now the case where $k = 1$. From part *(a)* and step 2, we have that

$$\phi_1(1) = \frac{n^1(1+\delta\tilde{\pi}^2)}{n^1 - \delta\tilde{\pi}^1}\frac{\theta^1}{c} + \frac{\delta^2\tilde{\pi}^1\tilde{\pi}^2 \cdot n^2}{(n^1 - \delta\tilde{\pi}^1)(n^2 - \delta\tilde{\pi}^2)}\frac{\theta^2}{c}$$

By definition, $s_1(2) = \frac{1-(\delta\tilde{\pi}^2 \cdot \frac{n^2-1}{n^2-\delta\tilde{\pi}^2})^2}{1-\delta\tilde{\pi}^2}$, so that (using $w_1(1) > w_2(1)$),

$$\frac{\phi_1(1)}{s_1(2)} > \frac{n^2(n^2 - \delta\tilde{\pi}^2)^2}{(n^2 - \delta\tilde{\pi}^2)(n^2-1)^2} > 1.$$

Again, we obtain the desired result in this standalone, residual case. Thus, there can be no equilibrium with finitely many senior cutpoints $\{s_1(k)\}_{k=1}^{\hat{k}}$ below ϕ_1, for any $\hat{k}$.

Step 6 (Conclusion). Step 5 rules out equilibria with finitely many senior cutpoints. Thus, any equilibrium must have infinitely many cut-points with

$$\overline{s}_1 = \lim_{k\to\infty} s_1(k) = \lim_{k\to\infty} \frac{1 - \left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^k}{1-\delta\tilde{\pi}^2}\frac{\theta^2}{c} = \left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c} < \phi_1.$$

Here, the second equality results from substituting the expression for $s_1(k)$ from (4.2), established in Step 2, and the final inequality is a direct consequence of Step 5. We now verify that the remaining candidate is, in fact, an equilibrium, and derive an explicit expression for ϕ_1. This completes the proof of the main proposition.

Step 3 uniquely identifies senior agent behavior as a cut-point strategy in equilibrium from any history h with $s^h < \overline{s}_1 < \phi_1$. Given the full completion ($\sigma^1(s) = s$) by junior agents from any $s \leq \phi_1$, it must be the case in any equilibrium— from Step 4— that $\sigma^2(s) = 0$ for all $s \in [\overline{s}_1, \phi_1]$. That is, the prescribed behavior of seniors is a best response to juniors' putative strategy $\sigma^1(s) = s$ for all $s \leq \phi_1$. It thus only remains to show there is no profitable deviation for juniors, given seniors' behavior.

Given that $\overline{s}_1 < \phi_1$, and $\sigma^2(s) = 0$ for all $s \in (\overline{s}_1, \phi_1]$, we have $V^1(\phi_1) = \tilde{\pi}^1\theta^1 + \tilde{\pi}^2\delta V^1(\phi_1) - \frac{\tilde{\pi}^1}{n^1}c(1-\delta)\phi_1$ and $V^2(\phi_1) = \tilde{\pi}^1\theta^2 + \tilde{\pi}^2\delta V^2(\phi_1)$, and thus we have

$$V^1(\phi_1) = \frac{\tilde{\pi}^1\theta^1 - \frac{\tilde{\pi}^1}{n^1}c(1-\delta)\phi_1}{1-\delta\tilde{\pi}^2} \quad \text{and} \quad V^2(\phi_1) = \left(\frac{1-\tilde{\pi}^2}{1-\delta\tilde{\pi}^2}\right)\theta^2 \tag{A.10}$$

Since in equilibrium juniors' values must verify $\delta V^1(\phi_1) = \theta^1 - c(1-\delta)\phi_1$, (A.10) directly gives (4.1). Substituting in (A.10) gives (4.5).

Next we show that junior agents don't have a profitable deviation from any h with s^h in $[0, \phi_1]$. At a history h' with $s^{h'} = \phi_1$, by definition junior agents are indifferent between completing the project or staying put. They clearly cannot gain by moving to a non-cutpoint point, so it is enough to consider deviations to senior agents' cutpoints. In equilibrium, $W^1(\phi_1, h') = \theta^1 - (1-\delta)c\phi_1$. A deviation to some $s_1(k)$, contributing $e' = \phi_1 - s_1(k)$, gives a payoff $W^1(e', h') = \delta V^1(s_1(k)) - (1-\delta)c(\phi_1 - s_1(k))$. This is not a profitable deviation iff

$$0 \geq \delta V^1(s_1(k)) + (1-\delta)cs_1(k) - \theta^1 \equiv \mathbb{L}_k^1 \tag{A.11}$$

Using now (A.7) we have that

$$\frac{\mathbb{L}_k^1}{1-\delta} = -\theta^1 + \frac{\delta}{1-\delta}\tilde{\pi}^2\mathbb{L}_{k-1}^1 - \delta\tilde{\pi}^2 cs_1(k-1) + \left(1 - \delta\frac{\tilde{\pi}^1}{n^1}\right) cs_1(k)$$

and replacing $s_1(k-1)$ with the expression from (A.6) we get

$$\begin{aligned}\frac{\mathbb{L}_k^1}{1-\delta} &= -\theta^1 + \frac{\delta}{1-\delta}\tilde{\pi}^2\mathbb{L}_{k-1}^1 + cw^2(1)\frac{n^2 - \delta\tilde{\pi}^2}{n^2-1} - cs_1(k)\left(\frac{1-\delta\tilde{\pi}^2}{n^2-1} + \delta\frac{\tilde{\pi}^1}{n^1}\right) \\ &\leq \frac{\delta}{1-\delta}\tilde{\pi}^2\mathbb{L}_{k-1}^1 + c\left(1-\delta\frac{\tilde{\pi}^1}{n^1}\right)\left(w^2(1) - w^1(1)\right)\end{aligned}$$

and the inequality is strict if $w^2(1) < s_1(k)$. Thus, juniors do not deviate to any $s_1(k)$ from arbitrary history h' with $s^{h'} = \phi_1$ if and only if $\mathbb{L}_1^1 \leq 0$, which is true since

$$\mathbb{L}_1^1 = (1-\delta)c(1-\delta\frac{\tilde{\pi}^1}{n^1})(w^2(1) - w^1(1)) < 0.$$

□

Proof of Theorem 4.5

We first prove the following lemma.

Lemma A.1. *For all histories h with $s^h > \phi_1$ and $m \in \{1, 2\}$, $\sigma^m(h) \leq s^h - \phi_1$.*

Proof of Lemma A.1. Consider first junior (group 1) agents. By construction, ϕ_1 is the largest s such that from any h with $s^h = s$, juniors finish the project outright, given $\sigma^2(h) = 0$. Since $\sigma^2(h) = 0$ minimizes free-riding incentives, ϕ_1 is the largest state corresponding to any history from which juniors would complete the project outright, for any σ^2. Thus, at any history h with $s^h > \phi_1$, $\sigma^1(h) < s^h$. By Lemma 4.1, from any h with $s^h \in (0, \phi_1)$, a junior agent strictly prefers to finish the project outright over playing $e \in [0, s^h)$. By the linearity of costs, it follows that for h with $s^h > \phi_1$ a junior agent would weakly prefer to finish the project outright over moving the project to $s' \in (0, \phi_1)$. Hence, at any history h with $s^h > \phi_1$, $\sigma^1(h) \leq s^h - \phi_1$. Next consider senior agents. Since σ^2 is a cutpoint strategy in $[0, \left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c})$, we know that for any k in the sequence, and any $s > s_1(k)$, $\sigma^2(s) \leq s - s_1(k)$. Thus, for all $s > \phi_1$, $\sigma^2(s) \leq s - \left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c}$. In equilibrium $\sigma^2(s) = 0$ for all $s \in [\left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c}), \phi_1]$. Thus, for any history h with $s^h > \phi_1$, moving the project to $s' \in [\left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c}), \phi_1)$ is dominated by moving the project to ϕ_1; this doesn't increase delay and reduces expected costs. Thus, for any h with $s^h > \phi_1$, $\sigma^2(h) \leq s^h - \phi_1$. □

Proof of Remark 4.6. For $\tau = 1$, $V^1(\phi_1) = (\alpha^1/\delta)\theta^1$, and for $2 \leq \tau \leq j(1)$: $V^1(\phi_\tau) = \alpha^1 V^1(\phi_{\tau-1}) \Rightarrow V^1(\phi_\tau) = (\alpha^1)^\tau \frac{1}{\delta}\theta^1$. On the other hand, for $\tau : 2 \leq \tau \leq j(1)^*$:

$$\phi_\tau - \phi_{\tau-1} = \left(\frac{1}{1 - \delta\tilde{\pi}^2 - \delta\tilde{\pi}^1/n^1}\right)\frac{\delta V^1(\phi_{\tau-1})}{c} = \phi_1(\alpha^1)^{\tau-1}$$

where we used (4.1) for $\tau = 1$. Summing up over all $2 \leq \tau \leq j(1)$ we get

$$\phi_{j(1)} = \phi_1\left[\sum_{\tau=0}^{j(1)-1}(\alpha^1)^\tau\right] = \phi_1\frac{1-(\alpha^1)^{j(1)}}{(1-\alpha^1)} \tag{A.12}$$

Thus, there is no alternation of effort on the equilibrium path iff $q < \phi_{j(1)}$. □

□

Proof of Proposition 4.8

Recall that $\Delta^m = \frac{\alpha^m}{\beta^m}$. In the text, we prove that if $\Delta^m < \Delta^{m'}$, then all stints of group m are one-step, except for potentially the first one, which is the case only if $m' = 1$. We prove the result for $\Delta^2 < \Delta^1$. The case of $\Delta^2 > \Delta^1$ is analogous. Let $\Omega^1(\phi_0) \equiv \frac{1-\delta\frac{\tilde{\pi}^2}{n^2}}{1-\delta\tilde{\pi}^1/n^1}\frac{\theta^1}{\theta^2}$. For $k-1$ even, (4.12) gives $\Omega^2(\phi_{J(k-1)}) = \Delta^2\Omega^2(\phi_{J(k-2)})$ and $\Delta^2 < \Omega^2(\phi_{J(k-1)}) < 1$, so

$$\Delta^2 < \Omega^1(\phi_{H(k-2)}) < 1. \tag{A.13}$$

And, for k odd, (4.12) gives $\Omega^1(\phi_{J(k)}) = [\Delta^1]^{j(k)}\Omega^1(\phi_{J(k-1)})$ and $\Delta^1 < \Omega^1(\phi_{J(k)}) < 1$

$$\implies \Delta^2\Delta^1 < [\Delta^1]^{j(k)}\Omega^1(\phi_{J(k-2)}) < \Delta^2. \tag{A.14}$$

Note that the left hand side of (A.14) and $\Omega^1(\phi_{J(k-2)}) < 1$ from (A.13) imply that $\Delta^2 < [\Delta^1]^{j(k)-1}$. Now consider $k-2$ which is odd and gives $\Delta^1 < \Omega^1(\phi_{J(k-2)}) < 1$. Thus, $\Delta^1 < \Delta^2\Omega^1(\phi_{J(k-1)})$ (since $\Omega^2(\phi_{J(k-1)}) = \Delta^2\Omega^2(\phi_{J(k-2)})$) and then $[\Delta^1]^{j(k)+1} < \Delta^2$ (since $[\Delta^1]^{j(k)}\Omega^1(\phi_{J(k-1)}) < 1$). Note that it must be that for all k odd

$$[\Delta^1]^{j(k)+1} < \Delta^2 < [\Delta^1]^{j(k)-1} \Leftrightarrow \frac{\log[\Delta^2]}{\log[\Delta^1]} - 1 < j(k) < \frac{\log[\Delta^2]}{\log[\Delta^1]} + 1$$

Assume first that $\log[\Delta^2] = \ell \log[\Delta^1]$ for some $\ell \in \mathbb{Z}^+$. Then we have that $\ell - 1 < j(k) < \ell + 1$ which implies that $j(k) = \ell$. Let now $\lfloor y \rfloor \equiv \sup\{x \in \mathbb{Z}_+ : x \leq y\}$, and write $\frac{\log[\Delta^2]}{\log[\Delta^1]} = \left\lfloor \frac{\log[\Delta^2]}{\log[\Delta^1]} \right\rfloor + \epsilon$ for some $\epsilon \in (0,1)$. It then follows that $\left\lfloor \frac{\log[\Delta^2]}{\log[\Delta^1]} \right\rfloor \leq j(k) \leq \left\lfloor \frac{\log[\Delta^2]}{\log[\Delta^1]} \right\rfloor + 1$.

□

Proof of Theorem 4.9 We prove the theorem for the case of public goods with an infinite-stream of payoffs θ^m in each period and then analogize to the case of a finite stream of payoffs of length T, which allows us to make the efficiency comparison as $\delta \to 1$. In particular, we prove the following result:

1. *Suppose $(\Delta^1)^y < \Delta^2 < \Delta^1$ for some $y \in \mathbb{N}$. Let y^* denote the smallest integer y^* such that this inequality holds. Then,*

$$\tilde{\phi} \leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2 - \frac{\tilde{\pi}^1}{n^1})}\left[\frac{1-(\alpha^1)^{j(1)}}{1-\alpha^1} + \frac{(\alpha^1)^{j(1)}\beta^2(1-(\alpha^1)^{y^*})}{(1-\alpha^1)(1-\beta^2(\alpha^1)^{y^*})}\right] + \frac{(\beta^1)^{j(1)}\theta^2}{c(1-\delta\tilde{\pi}^1 - \delta\frac{\tilde{\pi}^2}{n^2})(1-\alpha^2(\beta^1)^{y^*})}.$$

2. *Suppose $(\Delta^2)^y < \Delta^1 < \Delta^2$ for some $y \in \mathbb{N}$. Let y^* denote the smallest integer y^* such that this inequality holds. Then,*

$$\tilde{\phi} \leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2 - \frac{\tilde{\pi}^1}{n^1})}\left[\frac{1-(\alpha^1)^{j(1)}}{1-\alpha^1} + \frac{(\alpha^1)^{j(1)}(\beta^2)^{y^*}}{1-\alpha^1(\beta^2)^{y^*}}\right] + \frac{(\beta^1)^{j(1)}(1-(\alpha^2)^{y^*})\theta^2}{c(1-\delta\tilde{\pi}^1 - \delta\frac{\tilde{\pi}^2}{n^2})(1-\alpha^2)(1-\beta^1(\alpha^2)^{y^*})}.$$

3. *Suppose $\Delta^1 = \Delta^2 = \Delta$. Then*

$$\tilde{\phi} \leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2 - \frac{\tilde{\pi}^1}{n^1})}\left[\frac{1-(\alpha^1)^{j(1)}}{1-\alpha^1} + \frac{(\alpha^1)^{j(1)}\beta^2}{(1-\beta^2\alpha^1)}\right] + \frac{(\beta^1)^{j(1)}\theta^2}{c(1-\delta\tilde{\pi}^1 - \delta\frac{\tilde{\pi}^2}{n^2})(1-\alpha^2\beta^1)}.$$

Proof. We prove (1), as (2) is analogous. Part (3) follows directly from (1) or (2), making $y^* = 1$. From (4.1), we have

$$\begin{aligned}
\phi_1 &= \left(\frac{1}{1-\delta\tilde{\pi}^2-\delta\frac{\tilde{\pi}^1}{n^1}}\right)\frac{\theta^1}{c} \\
&\vdots \\
\phi_{j(1)} &= \left(\frac{1}{1-\delta\tilde{\pi}^2-\delta\frac{\tilde{\pi}^1}{n^1}}\right)\frac{\theta^1}{c}\sum_{\ell=1}^{h_1}\alpha^{\ell-1} = \left(\frac{1}{1-\delta\tilde{\pi}^2-\delta\frac{\tilde{\pi}^1}{n^1}}\right)\left(\frac{1-(\alpha^1)^{h_1}}{1-\alpha^1}\right)\frac{\theta^1}{c}.
\end{aligned}$$

At $\phi_{j(1)+1}$, the active contributor switches from group 1 to group 2. Hence,

$$\phi_{j(1)+1} = \phi_{j(1)} + \left(\frac{1}{1-\delta\tilde{\pi}^1-\delta\frac{\tilde{\pi}^2}{n^2}}\right)\frac{\delta V^2(\phi_{j(1)})}{c} = \phi_{j(1)} + \left(\frac{1}{1-\delta\tilde{\pi}^1-\delta\frac{\tilde{\pi}^2}{n^2}}\right)\frac{(\beta^1)^{j(1)}\theta^2}{c}.$$

In the case (1) being analyzed, $J(2) = j(1)+1$ since each stint of senior agents is of length one. Continuing forward,

$$\phi_{J(3)} = \phi_{J(2)} + \left(\frac{1}{1-\delta\tilde{\pi}^2-\delta\frac{\tilde{\pi}^1}{n^1}}\right)\frac{(\alpha^1)^{j(1)}\beta^2\theta^1}{c}\left(\frac{1-(\alpha^1)^{j(3)}}{1-\alpha^1}\right),$$

analogous to the formulation of $\phi_{j(1)}$. Moreover, note that by case (1), $j(3) \leq y^*$, and substituting gives us

$$\phi_{J(3)} \leq \phi_{J(1)} + \left(\frac{1}{1-\delta\tilde{\pi}^1-\delta\frac{\tilde{\pi}^2}{n^2}}\right)\frac{(\beta^1)^{j(1)}\theta^2}{c} + \left(\frac{1}{1-\delta\tilde{\pi}^2-\delta\frac{\tilde{\pi}^1}{n^1}}\right)\frac{(\alpha^1)^{j(1)}\beta^2\theta^1}{c}\left(\frac{1-(\alpha^1)^{y^*}}{1-\alpha^1}\right).$$

In general, note that the Lth stint for $L > 1$ odd will consist of at most y^* steps actively contributed by juniors, and the $(L+1)-$th one step stint by seniors. With this pattern, for any $L \geq 1$, the following formula holds for odd numbered stints:

$$\begin{aligned}
\phi_{J(2L+1)} \leq \phi_{J(1)} &+ \left(\frac{1}{1-\delta\tilde{\pi}^1-\delta\frac{\tilde{\pi}^2}{n^2}}\right)\frac{(\beta^1)^{j(1)}\theta^2}{c}\sum_{k=0}^{L-1}\left(\alpha^2(\beta^1)^{y^*}\right)^k + \\
&\left(\frac{1}{1-\delta\tilde{\pi}^2-\delta\frac{\tilde{\pi}^1}{n^1}}\right)\frac{(\alpha^1)^{j(1)}\beta^2\theta^1}{c}\left(\frac{1-(\alpha^1)^{y^*}}{1-\alpha^1}\right)\sum_{k=0}^{L-1}\left(\beta^2(\alpha^1)^{y^*}\right)^k. \quad \text{(A.15)}
\end{aligned}$$

Substituting for $\phi_{J(1)}$ from above and taking the limit as $L \to \infty$ in (A.15),

$$\tilde{\phi} \leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2 - \frac{\tilde{\pi}^1}{n^1})}\left[\frac{1-(\alpha^1)^{j(1)}}{1-\alpha^1} + \frac{(\alpha^1)^{j(1)}\beta^2(1-(\alpha^1)^{y^*})}{(1-\alpha^1)(1-\beta^2(\alpha^1)^{y^*})}\right] + \frac{(\beta^1)^{j(1)}\theta^2}{c(1-\delta\tilde{\pi}^1 - \delta\frac{\tilde{\pi}^2}{n^2})(1-\alpha^2(\beta^1)^{y^*})} \tag{A.16}$$

There are two possible cases. For brevity, we consider the case in which there exists a minimal positive integer y^* such that $(\Delta^1)^{y^*} < \Delta^2 < \Delta^1$ (the case in which $(\Delta^2)^{y^*} < \Delta^1 < \Delta^2$ for some integer y^* is analogous). From (A.16),

$$\begin{aligned}\tilde{\phi} &\leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2 - \delta\frac{\tilde{\pi}^1}{n^1})}\left[\frac{1}{1-\alpha^1} - \frac{(\alpha^1)^{j(1)}(1-\beta^2)}{(1-\alpha^1)(1-\beta^2(\alpha^1)^{y^*})}\right] + \frac{(\beta^1)^{j(1)}\theta^2}{c(1-\delta\tilde{\pi}^1 - \delta\frac{\tilde{\pi}^2}{n^2})(1-\alpha^2(\beta^1)^{y^*})} \\ &< \frac{\theta^1}{c(1-\delta\tilde{\pi}^2 - \delta\frac{\tilde{\pi}^1}{n^1})(1-\alpha^1)} + \frac{\theta^2}{c(1-\delta\tilde{\pi}^1 - \delta\frac{\tilde{\pi}^2}{n^2})(1-\alpha^2)},\end{aligned}$$

where the second inequality follows because $\beta^1 < 1$. Now, for each group $m \in \{1, 2\}$,

$$1-\alpha^m = 1 - \frac{(1-1/n^m)\tilde{\pi}^m\delta}{1-\delta\tilde{\pi}^{-m} - \delta\frac{\tilde{\pi}^m}{n^m}} = \frac{1-\delta}{1-\delta\tilde{\pi}^{-m} - \delta\frac{\tilde{\pi}^m}{n^m}}$$

Substituting into the above inequality, and re-arranging we obtain $c\tilde{\phi} < \frac{\theta^1+\theta^2}{1-\delta}$ for any fixed $\delta < 1$, in the case of infinite-lasting flow payoffs. For a project whose flow payoffs last exactly T periods, the analogous expression is

$$c\tilde{\phi} < \frac{(1-\delta^T)(\theta^1+\theta^2)}{1-\delta} = \hat{\theta}^1 + \hat{\theta}^2.$$

Since $c\tilde{\phi} < \hat{\theta}^1 + \hat{\theta}^2$ for any $\delta < 1$, then for $n^m > 1$ for some $m \in \{1, 2\}$, $\lim_{\delta \to 1} c\tilde{\phi} \leq \hat{\theta}^1 + \hat{\theta}^2 < n^1\hat{\theta}^1 + n^2\hat{\theta}^2$. The second result follows since $\tilde{\phi}$ is the project size completed in equilibrium.

□

Proof of Proposition 4.10

We show that the expected completion time of a project that requires L contribution stints, $\{h_1, \ldots, h_L\}$, is given by

$$\mathcal{E}\left(\{j(\ell)\}_{\ell=1}^L\right) = \sum_{\ell=1}^L j(\ell) + \sum_{\ell=1}^L \tilde{\pi}_{m_{-\ell}}(\tilde{\pi}_{m_\ell})^{j(\ell)-2}, \tag{A.17}$$

where $m_\ell = 1$ if ℓ is odd and 2 if ℓ is even, and $m_{-\ell} = m \in M \neq m_\ell$. The second part of the proposition follows by the argument in the text. Consider the sequence $\{j(\ell)\}_{\ell=1}^{L}$ and define the random variable $x\left(\{j(\ell)\}_{\ell=1}^{L}\right)$, as the number of periods it takes to finish the project. Note that for $k = 0, 1, 2, \ldots$

$$x\left(\{j(\ell)\}_{\ell=1}^{L}\right) = j(L) + k + x\left(\{j(\ell)\}_{\ell=1}^{L-1}\right) \qquad \text{w.p.} \qquad (\tilde{\pi}^1)^{j(L)}(\tilde{\pi}^2)^k$$

It follows that $\mathbb{E}\left(x\left(\{j(\ell)\}_{\ell=1}^{L}\right)\right) = \sum_{\ell=1}^{L} \mathbb{E}\left(x\left(j(\ell)\right)\right)$. Letting $m_\ell = 1$ if ℓ is odd and 2 if ℓ is even, and $m_{-\ell} = m \in M \neq m_\ell$, we have:

$$\mathbb{E}\left(x\left(j(\ell)\right)\right) = j(\ell) + (\tilde{\pi}_{m_\ell})^{j(\ell)}\left[\sum_{k=1}^{\infty}(\tilde{\pi}_{m_{-\ell}})^k k\right] = j(\ell) + (\tilde{\pi}_{m_\ell})^{j(\ell)-2}\left[\tilde{\pi}_{m_{-\ell}}\right],$$

$$\Rightarrow \mathbb{E}\left(x\left(\{j(\ell)\}_{\ell=1}^{L}\right)\right) = \sum_{\ell=1}^{L} j(\ell) + \sum_{\ell=1}^{L}(\tilde{\pi}_{m_\ell})^{j(\ell)-1}\left[\frac{\tilde{\pi}_{m_{-\ell}}}{\tilde{\pi}_{m_\ell}}\right]$$

Now, suppose $q = \phi_\ell$ for $\ell > j(1)$ and $\Delta^m > \Delta^{m'}$, and let $\mathcal{E}_1 \equiv j(1) + \tilde{\pi}^2(\tilde{\pi}^1)^{j(1)-2}$, as in the statement of the proposition. Since $\Delta^m > \Delta^{m'}$, before the final stint of $j(1)$ small projects by juniors, juniors and seniors alternate in a contribution cycle in which group m' completes single small projects, while group m completes either $x^m \equiv \left\lfloor \frac{\log[\Delta^{m'}]}{\log[\Delta^m]} \right\rfloor$ or $x^m + 1$ small projects before turning it over to members of the other group. We can then split the terms of (A.17) in stints taken by each group, and incorporate this information to compute expected delay for a project of size q.

We show the result for $\hat{k}(q)$ odd, noting that the even case is analogous. In addition to the last stint by group 1 agents there are $(\hat{k}(q) - 1)/2$ stints by each group, so for some $x \in [x^m, x^m + 1]$,

$$\mathcal{E}\left(\{j(\ell)\}_{\ell=1}^{\hat{k}(q)}\right) = \mathcal{E}_1 + \left(\frac{\hat{k}(q) - 1}{2}\right)\left\{(1 + x) + \frac{\tilde{\pi}^m}{\tilde{\pi}^{m'}} + \tilde{\pi}^{m'}(\tilde{\pi}^m)^{x-2}\right\}.$$

In particular, if $\Delta^m = \Delta^{m'}$, groups alternate in single-step stints, so

$$\mathcal{E}\left(\{j(\ell)\}_{\ell=1}^{\hat{k}(q)}\right) = \mathcal{E}_1 + \frac{\hat{k}(q) - 1}{2\tilde{\pi}^1\tilde{\pi}^2}.$$

□